ELEMENTARY MECHANICS

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PREFACE

THE present volume is intended to take the place of my previous book entitled "The Elements of Applied Mathematics." It includes, however, only Dynamics and Statics, the Hydrostatical portion of the earlier work being now issued separately.

The revision has been carried out by my colleague Dr T. H. Havelock, who has re-arranged and largely re-written the subject-matter; the scope of the former book has been enlarged by the addition of a chapter on "The Energy of Rotating Bodies," while sections on Bending Moment, Simple Harmonic Motion, &c., have been included, and many fresh examples inserted.

It is hoped that the book in its present form will be found to deal adequately with all those portions of Elementary Mechanics which can be studied with advantage without the aid of the Differential Calculus.

C. M. JESSOP.

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ELEMENTARY MECHANICS

CHAPTER I.

MOTION IN A STRAIGHT LINE.

1. THE ideas which form the basis of our subject are those of *space* and *time*, which, without attempting to define, we shall regard as primary notions.

2. Another idea which is fundamental is that of *matter*.

Matter may be defined as that which affects our senses so as to produce such ideas as *resistance, size, shape,* &c.*

The first of these, namely resistance, is that which is most closely connected with Mechanics; in this respect, matter is said to possess the property of *inertia*.

The measure of this property for any given body is called the *mass* of the body. The masses of two or more bodies may be compared in various ways such as weighing, or lifting.

3. The Measure of a quantity.

In what follows we shall have occasion to use the word *measure*. The measure of any quantity such as a length, a time &c., is the number of times it contains that quantity of the *same* kind which we have agreed to call the *unit*. Thus the measure of a line 6 inches long is 12 if we agree that half-an-inch shall be the unit of length. If the unit of time is one second then the measure of half-an-hour is 1800, and so on.

* 'Whatever can occupy space' is sometimes given as the definition of matter. Tait, *Properties of Matter*.

4. Units of length.

There are two units of length in common use, one is the *foot*, the other, which is used mainly in scientific measurements, is the *centimetre*.

A centimetre is the $\frac{1}{100}$ part of a *metre*, a metre being 39·37 inches. It is the length of a certain bar of platinum, kept at Paris, when at the temperature 0° C.

5. Velocity.

The velocity of a moving point is its rate of motion, or its rate of change of position. Thus velocity may be of any magnitude and be in any direction. In the present chapter we shall only consider the motion of a point which always moves in the *same straight line*.

When in what follows we speak of the motion of a *body*, the motion considered is one of *translation*, i.e. one in which the velocity of each particle of the body is the same at the same instant, in other words the body moves without *rotation*. The velocity of the body is then that of any one of its particles.

6. Uniform velocity.

The velocity of a body is said to be uniform when it moves over equal distances in equal times, whatever those times may be. It is measured by the space moved over in the unit of time.

Velocity is said to be *variable*, when distances passed over in equal times are not equal.

It is measured at any instant, by the distance which would be passed over by the body in a unit of time if the velocity were to remain the same as at the given instant for a unit of time.

The more advanced student will be able to see that this definition of variable velocity may be replaced by the following: if s is the measure of the space described in t seconds after the given instant, then, when s and t are very small, the velocity of the body is the *limit* of the fraction $\frac{s}{t}$.

7. Unit of velocity.

The unit of velocity is the velocity of a body which moves over a unit of length in a unit of time. A body

is said to have a velocity 10 (or generally v), when 10 (or v) units of length are passed over in the unit of time.

The units of space and time are generally a foot and a second. These are sometimes called "standard" units.

We shall call the velocity of one foot per second *one foot-second.*

A velocity of one centimetre per second is called a *kine*.

8. Space described in t seconds.

The foot and the second being the units of length and time a body which moves uniformly with a velocity v, passes over v feet each second, and therefore vt feet in t seconds.

Hence if s is the number of feet described in t seconds

$$s = vt, \text{ and } v = \frac{s}{t}.$$

9. Average velocity.

The *average* velocity of a body whose actual velocity is variable is that velocity which would have carried the body through the distance actually described in the same time at a uniform rate.

For instance, suppose a body to pass over 2, 3, 4, 5 feet respectively in 4 consecutive seconds. The entire distance is 14 feet. The average velocity is $3\frac{1}{2}$ feet per second.

If the velocity is uniform the average velocity is the same as the actual velocity.

Ex. 1. A point moves with a uniform velocity of 12 feet per second, find the whole space described in 3 minutes.

Twelve feet are passed over each second, therefore in three minutes, or 180 seconds, 180 times that distance will be described. The answer is then 12×180 feet, or 720 yards.

Ex. 2. A point moves for 10 seconds with an average velocity of 5 feet per second; during the first part of the time the velocity is 3 feet per second, and during the latter part it is 7 feet per second; what is the length of this part?

Let s be the space described in 10 seconds; thus $s = 50$ feet.

Let s_1 and s_2 denote the spaces described in the first and second intervals respectively of t_1 and t_2 seconds,

then $$\frac{s_1}{t_1} = 3,$$

and $$\frac{s_2}{t_2} = 7;$$

hence $$3t_1 + 7t_2 = s_1 + s_2 = s = 50,$$

and $$t_1 + t_2 = 10,$$

from which it follows that

$$t_2 = 5 \text{ secs.}, \therefore s_2 = 35 \text{ feet}.$$

Ex. 3. A body describes 6 feet in 4 seconds; if the unit of time be two minutes, what must be the unit of length in order that the measure of the velocity may be unity?

Since 6 feet are described in 4 seconds, $\frac{6}{4}$ feet are described in one second, and $\frac{6}{4} \times 120$ feet in two minutes which is here the unit of time. In order that the measure of this velocity may be unity the distance described in the unit of time must be the unit of length; the unit of length is therefore $\frac{6}{4} \times 120$ feet or 60 yards.

EXAMPLES. I.

1. A train moving uniformly goes 300 miles in 5 hours; find its velocity in feet per second.

2. How long would a train take to go 220 yards if moving at the rate of 30 miles an hour?

3. A train takes 2 hours and 20 minutes to go 115 miles; find its average velocity.

4. Assuming the velocity of sound to be 1100 feet per second, find the distance of the point of discharge, if 24 seconds elapse between seeing the lightning and hearing the thunder.

5. A train travels 205 miles in 7 hrs. 31 minutes; find its average velocity in feet per second.

6. How long does it take light to travel from the sun to the earth, the distance of the sun being 91,000,000 miles, and the velocity of light 186,000 miles per second?

7. A gun is fired on board a ship at sea, an echo is heard from a cliff after a lapse of 19·2 seconds; find the distance of the cliff, using the velocity of sound given above.

10. Velocities are represented by lines.

Since a velocity is completely determined by its direction and magnitude it can be represented by a *straight line*, for a line can be drawn

(i) in any direction, and therefore in the direction of the velocity;

(ii) of any length, and therefore of a length to represent the magnitude of the velocity, the line containing as many units of length, as the velocity contains units of velocity.

11. Acceleration.

A body is said to have *uniform acceleration* of its velocity, when equal changes of velocity occur in equal intervals of time, whatever those times may be.

It is measured by the number of units of velocity added or subtracted in each unit of time.

In the latter case the velocity is diminished, and the speed of the body is therefore retarded; the rate of change, however, is still termed an acceleration, but it has a *negative* value.

Variable acceleration occurs when in equal intervals of time unequal changes of velocity occur; for the present we consider only uniform acceleration.

12. Unit of acceleration.

The unit of acceleration, is the acceleration of a body which moves so that its velocity is increased by the unit of velocity in each unit of time.

When a body is said to have an acceleration 32, or generally a, the meaning is that its velocity increases by 32, or a, units of velocity in each unit of time. If the units of length and time are a foot and a second, a body whose velocity is increased every second by a units of velocity, is said to have an acceleration of a foot-second units.

For instance, it will be seen subsequently that a falling body has an acceleration of 32 foot-second units. This means that 32 units of velocity are gained each second of the body's

motion, a unit of velocity being a velocity of one foot per second, or a foot-second.

A change of velocity like a velocity is completely determined by its direction and magnitude, hence *accelerations can be represented by lines.*

13. To prove that $v = u + at$.

To find the velocity after a time t of a body moving with constant acceleration a.

The velocity is increased each second by a units of velocity.

Hence at the end of the first second the velocity is a,
.... next............................$2a$,
.... third...........................$3a$,
and so on.

Thus at the end of t seconds the velocity is at; denoting the velocity after t seconds by v we therefore have

$$v = at \quad \text{.........................(i)}.$$

If the body had at first a velocity u, or an *initial velocity* u, we see in like manner that

$$v = u + at \quad \text{.......................(ii)}.$$

If the direction of the acceleration is opposite to that of the initial velocity, in t seconds at units of velocity will be *lost*, thus in this case

$$v = u - at \quad \text{......................(iii)}.$$

Ex. 1. A body starting with a velocity of 4 feet per second has an acceleration 3, find its velocity after 5 seconds.

From (ii) we see that

$$v = 4 + 3 \times 5 = 19,$$

hence the required velocity is 19 feet per second.

Ex. 2. A body starting with a velocity of 128 feet per second has an acceleration 32 in the direction opposite to this initial velocity, when will the body be brought to rest?

After t seconds we have

$$v = 128 - 32t,$$

when v is zero $\quad 0 = 128 - 32t,$

or $\quad t = 4$ seconds.

Ex. 3. A train having a uniform acceleration starts from rest and at the end of 3 seconds has a velocity with which it could travel through one mile in the next 5 minutes. Find its acceleration.

If a be the acceleration, the velocity at the end of 3 seconds is $3a$ feet per second. With this velocity the train would pass over one mile in 5 minutes, or 1760×3 feet in 5×60 seconds.

$$\therefore 3a = \frac{1760 \times 3}{5 \times 60};$$

hence $a = 5{\cdot}8\dot{6}$ foot-secs. per second.

EXAMPLES. II.

1. A body starting with a velocity of 90 feet per second after 5 seconds has a velocity of 50 feet per second; what is its acceleration?

2. A body whose initial velocity is 10 feet per second and final velocity 50 feet per second, has an acceleration of 10 ft.-sec. units; in what time will it gain this final velocity?

3. A point starting from rest, after 6 seconds has a velocity of 18 feet per second; find its acceleration.

4. A point whose initial velocity is 20 feet per second, has an acceleration of 32 ft.-sec. units; find its velocity after 5 seconds.

5. The initial velocity of a body is 11 feet per second, and the body's velocity is increased each second by a velocity of 7 feet per second; find when it will be moving at the rate of 60 miles an hour.

6. A body has a velocity of 9 feet per second at the beginning, and of 10 feet per second at the end of a given second; what will be its velocity after 5 more seconds?

14. Representation by an area of the space described.

On a given straight line OA measure off a line OP containing t units of length, and on a perpendicular line OB measure off a line OQ containing v units of length.

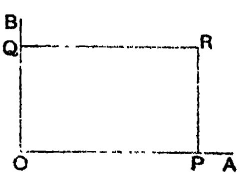

FIG. 1.

Complete the rectangle $OPRQ$. The area of this rectangle is $OP \times OQ$ or vt, hence since $vt = s$, the area of the rectangle measures the space described by a body moving for t seconds with a uniform velocity v.

15. Lines representing the velocity of a moving point.

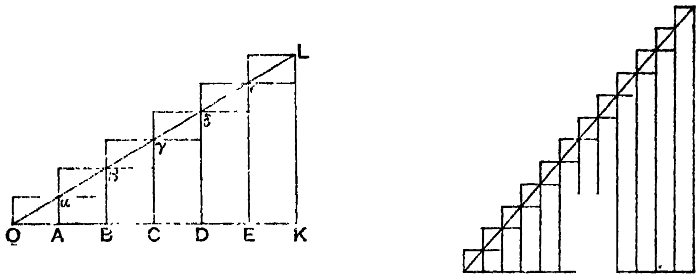

Fig. 2.

Let the line OK be of length t and be divided into equal portions OA, AB, BC, CD &c.

Draw the vertical line $A\alpha$ to represent the velocity of the point at the end of the first portion of time. Join $O\alpha$ and produce it, drawing verticals through B, C, ... to meet it at β, γ, ...

Then we easily see that

$$B\beta = 2A\alpha, \quad C\gamma = 3A\alpha, \text{ \&c.}$$

Thus the lines $A\alpha$, $B\beta$, $C\gamma$ ... represent the velocities of the point at the ends of the first, second, third ... portions of the time, KL representing its final velocity or at.

16. To prove that $s = \frac{1}{2}at^2$.

In the figure of last Article draw horizontal and vertical lines through α, β, γ ... forming two sets of rectangles, one within and the other without the triangle OKL.

Consider the rectangles on the base BC. The smaller one measures the space described by a point in a time represented by BC, if moving with constant velocity $B\beta$, Art. 14; or a *smaller space* than that actually described by the point since its velocity during the time BC is greater than $B\beta$.

Again the larger rectangle measures the space which would be described if the point moved with constant ve-

locity $C\gamma$, during the time BC, or a *greater space* than the actual one since the point's velocity is less than $C\gamma$ during the time BC. Hence the true space described by the point in the time BC is between these values. During each of the times represented by OA, AB, BC ... a similar result is true. We have therefore that the

measure of the sum of the inner rectangles is $<$ space described,
.............................outer ....$>$ space described.

Now if we divide OK into a very great number of equal parts, the sum of each set of rectangles will *approach* the area of the triangle OKL very closely, as we see from the figure on the right.

In fact the difference of each sum from the area of OKL is equal to $\frac{1}{2}KL \times OA$ which becomes indefinitely small as we increase the number of divisions.

The measure of the space actually described is therefore the area of the triangle

$$OKL = \tfrac{1}{2}OK \times KL = \tfrac{1}{2}t \times at = \tfrac{1}{2}at^2.$$

Thus $$s = \tfrac{1}{2}at^2.$$

If the body has initially a velocity u, draw OM perpendicular to OK and of length u, complete the rectangle $OMNK$.

In this case the velocity of the body at each instant is greater than its former value by u, and the space described will be measured by the sum of the areas OLK and $OMNK$, that is

$$s = \tfrac{1}{2}at^2 + ut.$$

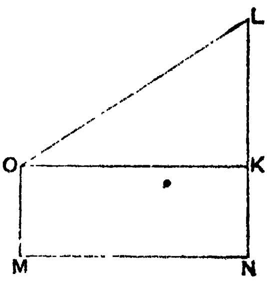

FIG. 3.

17. Alternative proof for the formula

$$\mathbf{s = ut + \tfrac{1}{2}at^2}.$$

To find the space passed over in the time t by a body moving with an acceleration a, and possessing an initial velocity u, we may also adopt the following method.

Divide the given time t into n equal intervals, each of length $\frac{t}{n}$; the velocities at the beginnings of the first, second, ... nth of these intervals are

$$u,\ u+a\frac{t}{n},\ \ u+2a\frac{t}{n},\ \ldots,\ u+(n-1)a\frac{t}{n};$$

since a velocity $a\frac{t}{n}$ is added during each interval, Art. 12.

Now if the body moved *uniformly* during each interval with the velocity which it actually has at the beginning of that interval, each space so described would be less than the space actually described in that interval.

And the sum of the spaces described in this supposed manner is

$$u\frac{t}{n}+\left(u+a\frac{t}{n}\right)\frac{t}{n}+\left(u+2a\frac{t}{n}\right)\frac{t}{n}+\ldots$$

$$+\left(u+(n-1)a\frac{t}{n}\right)\frac{t}{n} \qquad \text{Art. 8,}$$

$$=n\frac{ut}{n}+\frac{at^2}{n^2}(1+2+3+\ldots+\overline{n-1})$$

$$=ut+\frac{at^2}{n^2}\frac{n(n-1)}{2}\left[\text{since } 1+2+3+\ldots+n-1=\frac{n(n-1)}{2}\right]$$

$$=ut+\frac{1}{2}at^2\left(1-\frac{1}{n}\right).$$

Again, if the body moved uniformly throughout each interval with the velocity it has at the end of that interval, each space so described would be greater than the space actually described in that interval.

The sum of the spaces described in this manner is

$$\left(u+a\frac{t}{n}\right)\frac{t}{n}+\left(u+2a\frac{t}{n}\right)\frac{t}{n}+\ldots+\left(u+na\frac{t}{n}\right)\frac{t}{n}$$

$$=nu\frac{t}{n}+\frac{at^2}{n^2}(1+2+\ldots+n)$$

$$= ut + \frac{at^2}{n^2}\frac{n(n+1)}{2}$$

$$= ut + \tfrac{1}{2}at^2\left(1 + \frac{1}{n}\right).$$

The space actually described lies therefore between

$$ut + \tfrac{1}{2}at^2\left(1 - \frac{1}{n}\right) \text{ and } ut + \tfrac{1}{2}at^2\left(1 + \frac{1}{n}\right).$$

By making the intervals *small* enough, and therefore n *large* enough, we can make $\frac{1}{n}$ as small as we please, thus causing these two expressions to continually approach each other in value. The actual space s described is therefore

$$ut + \tfrac{1}{2}at^2 \text{ ....(iv)},$$

as this is the limiting value to which each expression approaches as n increases.

18. If the direction of the acceleration is contrary to that of the initial velocity u, we get in the same way,

$$s = ut - \tfrac{1}{2}at^2.$$

N.B. If the body started from rest or with initial velocity zero, putting u equal to zero in the formula just proved we obtain

$$s = \tfrac{1}{2}at^2 \text{ .... (v)}.$$

We could of course have got this value for s by going through the proof of the last Article omitting u throughout.

Ex. 1. A body has an initial velocity of 100 feet per second, and moves with an acceleration of 10 ft.-sec. units; what is the distance passed over in 5 seconds?

If s be the required distance

$$s = 100 \times 5 + \tfrac{1}{2} \times 10 \times 5^2$$
$$= 625 \text{ feet.}$$

Ex. 2. The initial velocity of a body is 40, and it moves with an acceleration of -2; find when it will be 400 feet from the starting-point.

We have $400 = 40t - t^2$.

Solving this quadratic equation we obtain 20 as the required number of seconds.

Ex. 3. A body starts from rest and moves with a uniform acceleration of 18 ft.-sec. units. Find the time required by it to traverse the first, second, and third foot respectively.

For the first foot we have

$$1=9t^2, \quad \therefore \quad t=\tfrac{1}{3} \text{ of a sec.}$$

The time required to traverse the second foot is that taken to traverse 2 feet—(the time taken to traverse 1 foot), hence it is

$$\frac{\sqrt{2}}{3}-\frac{1}{3}=\frac{1}{3}(\sqrt{2}-1) \text{ secs.}$$

Similarly we find as the time for the third foot.

$$\frac{1}{\sqrt{3}}-\frac{\sqrt{2}}{3}=\frac{\sqrt{3}-\sqrt{2}}{3} \text{ feet.}$$

EXAMPLES. III.

1. With an acceleration of 32 ft.-sec. units, how far will a particle move in 10 seconds starting from rest, and what will be its velocity at the end of that time?

2. A particle moves with uniformly increasing velocity. Show that the whole space is proportional to the square of the whole time.

3. A body moves over 3 ft., 5 ft., 7 ft., 9 ft. respectively, in 4 consecutive seconds; find its average velocity.

4. A body has an initial velocity of 20 feet per second, and a positive acceleration of 32 ft.-sec. units; how long will it take to pass over 1800 feet?

5. Starting with a velocity of 200 centimetres per second, a body has an acceleration of 2 centimetre second units. Find when its velocity is zero and how far it has gone in the time.

6. In what time will a body acquire a velocity of 60 miles an hour, if it starts with a velocity of 28 feet per second, and moves with the foot-sec. unit of acceleration?

19. To show that $v^2 = u^2 + 2as$.

We have seen that $\qquad v = u + at,$

and that $\qquad s = ut + \frac{1}{2}at^2.$

From the first equation

$$v^2 = u^2 + 2uat + a^2t^2,$$

or $$v^2 = u^2 + 2a(ut + \tfrac{1}{2}at^2),$$

hence from the second

$$v^2 = u^2 + 2as \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(vi)}.$$

Ex. 1. A body has an initial velocity of 6 feet per second and moves over a distance of 4 feet with an acceleration of 8 foot-secs. per second; what is the final velocity?

Here
$$v^2 = 6^2 + 2 \times 8 \times 4,$$
$$= 36 + 64 = 100;$$
$$\therefore\ v = 10 \text{ feet per second.}$$

Ex. 2. If the velocity of a body increase from 12 feet to 13 feet per second while it moves over a distance of 5 feet, what is the acceleration?

Applying the formula,
$$13^2 = 12^2 + 10a;$$
we have that
$$a = \frac{13^2 - 12^2}{10} = 2\cdot5.$$

Ex. 3. The speed of a railway train increases uniformly for the first three minutes after starting, and during this time it travels one mile. What speed in miles per hour has it now gained, and what space did it describe in the first two minutes?

Since the train passes over one mile in three minutes we have

$$3 \times 1760 = \frac{a}{2} \times (3 \times 60)^2;$$

$$\therefore\ a = \frac{3 \times 1760 \times 2}{(3 \times 60)^2} = \frac{44}{135} \text{ ft.-sec. units.}$$

The velocity gained in passing over a mile is by Art. 19

$$\sqrt{2 \times \tfrac{44}{135} \times 3 \times 1760},$$
$$= \tfrac{176}{3} \text{ feet per second, or 40 miles per hour.}$$

The space described in the first two minutes

$$= \tfrac{1}{2} \times \tfrac{44}{135} \times (2 \times 60)^2$$
$$= \tfrac{7040}{3} \text{ feet}$$
$$= \tfrac{4}{9} \text{ of a mile.}$$

EXAMPLES. IV.

Ex. 1. A railway train whose mass is 100 tons moving at the rate of a mile a minute, is brought to rest in 10 seconds by the action of a uniform force. Find how far the train runs in the time during which the force is applied.

Ex. 2. A body whose initial velocity is 30 feet per second, passes over a distance of 50 feet and has then a velocity of 60 feet per second, what is the acceleration?

Ex. 3. A body which has an acceleration of -32 ft.-sec. units has an initial velocity of 32 feet per second; find how far it will go before coming to rest, and in what time it will do so.

Ex. 4. A body moves for 6 seconds with a constant acceleration during which time it describes 81 feet, the acceleration then ceases, and during the next 6 seconds it describes 72 feet; find its initial velocity and its acceleration.

20. Recapitulation.

Collecting the results of the last three Articles we see that

if the body starts from rest

$$v = at \qquad (1),$$

$$s = \frac{v}{2}t \qquad (2).$$

$$s = \tfrac{1}{2}at^2 \qquad (3),$$

$$v^2 = 2as \qquad (4),$$

or

(1) the velocity gained in t seconds is t times the velocity gained in one second,

(2) the distance passed over in t seconds is t times the average distance passed over in one second,

(3) the space described varies as the square of the time,

(4) the space described varies as the square of the velocity.

If the body has an initial velocity u

$$v = u + at \qquad (5),$$

$$s = \frac{u+v}{2}t \qquad (6),$$

$$s = ut + \tfrac{1}{2}at^2 \qquad (7),$$

$$v^2 = u^2 + 2as \qquad (8).$$

21. Velocity-time curve.

Consider a particle moving in a straight line with a velocity which varies in any manner but without any sudden changes. Suppose we can measure the velocity at any given instant, that is, the average velocity during a very short interval of time including the instant in question (§ 6). Let us choose a scale on which a unit of length is to represent one second, and mark on a base line OX points representing given instants; thus O represents the instant from which the time is measured and, if ON is t units of length on the scale, N represents an instant t seconds later. We choose another (or the same) scale such that unit length represents a velocity of one foot per second; from N draw a perpendicular NP representing the velocity of the particle at the instant given by N. We obtain thus a point P, and if we perform the operations for a sufficient number of instants we obtain a series of points which can be joined by a continuous curve. This curve is called the velocity-time curve; it represents graphically the change of the velocity with the time.

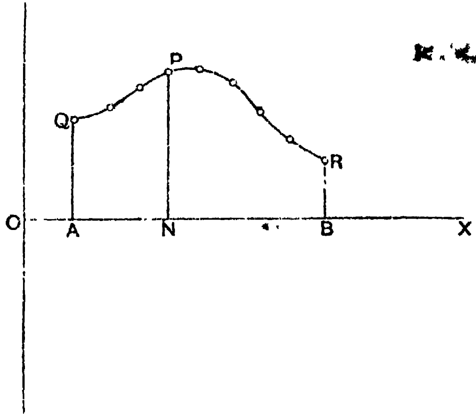

FIG. 4.

Now acceleration is the rate of change of velocity with the time, so we see that the acceleration at any instant N depends upon the steepness, or slope, of the velocity-time curve at P; thus fig. 4 must refer to a particle whose acceleration is not uniform but varies in magnitude, being sometimes positive and at other times negative. If we know the law according to which the particle is moving we can find accurately the form of the velocity-time curve. For instance, we have seen that if the particle is moving with *uniform* acceleration, the velocity-time curve is a straight line, that is, a line of constant slope. Let fig. 5 represent this case. Then since the acceleration a is constant, we have from the definition of acceleration

$$a = \frac{v - u}{t}.$$

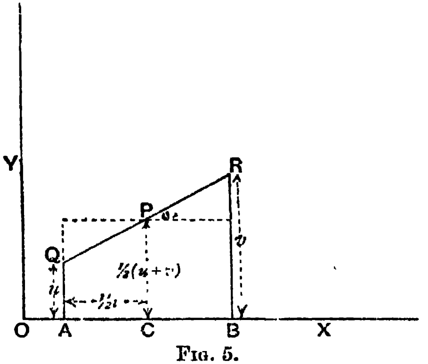

FIG. 5.

We have seen that the space described is represented by the area $AQRB$; hence the average velocity in the interval t is equal to $\frac{1}{2}(u+v)$. In this case the *average* velocity during the interval is the *actual* velocity at the middle *instant* of the interval.

Hence we have, for uniform acceleration,

$$s = (\text{average velocity}) \times (\text{time}) = \tfrac{1}{2}(u+v)\,t = ut + \tfrac{1}{2}at^2$$

$$= \tfrac{1}{2}(u+v) \times \frac{v-u}{a} = \frac{v^2-u^2}{2a}.$$

It can be shown further that in general for fig. 4 the space described is represented by the area enclosed by the velocity-time curve, the base line AB, and the initial and final ordinates QA and RB. But in general the average velocity is not the arithmetic mean of the initial and final velocities, nor is it the actual velocity at the middle instant; these statements hold only for motion with constant acceleration.

22. Space described in a given second.

We find the space described in the nth second after the start, when the body begins with velocity u and moves with constant acceleration a, as follows: since the space described equals (average velocity) $\times$ (time), the space described in the nth second is equal numerically to the average velocity during the nth second; but the acceleration is constant, hence the average velocity is the actual velocity at the middle of the nth second, that is, the actual velocity at $(n - \frac{1}{2})$ seconds after the start, or

$$u + (n - \tfrac{1}{2})\,a.$$

Hence the spaces described in the first, second, third, .. nth seconds are respectively

$$u + \tfrac{1}{2}a, \quad u + \tfrac{3}{2}a, \quad u + \tfrac{5}{2}a, \ldots, \quad u + (n - \tfrac{1}{2})\,a.$$

22 *a* If the distance s in feet covered by a particle from rest in t seconds is given by the equation $s = 9t^2$, to estimate the velocity of the particle 4 seconds after the start.

Space covered in 4 seconds $= s = 9\,(4)^2 = 144$ ft

... . $(4 + T)$ seconds

$$s' = 9\,(4 + T)^2 = (144 + 72T + 9T^2) \text{ ft}$$

Hence the space covered in an interval T beginning with the instant in question $(t = 4) = s' - s = 72T + 9T^2$.

Average velocity during the interval T

$$= \frac{s' - s}{T} = (72 + 9T) \text{ ft per sec.}$$

Hence the average velocity during

$\frac{1}{10}$th sec. after the instant $t = 4$ is 72·9 ft per sec.

$\frac{1}{100}$th sec................................72·09.............

$\frac{1}{1000}$th sec ....72·009 ....

....

By taking T' smaller and smaller, we see that the average velocity during the interval approaches more and more closely to the value 72. But when the velocity is variable, the velocity at any instant is defined as the limiting value of the average velocity during a short interval beginning with the instant in question. Hence in this case the velocity at 4 seconds after the start is 72 feet per second.

Ex. For the above example, estimate the velocity at the end of 1, 2, 3, 4 and 5 seconds from the start, and draw the velocity-time curve.

EXAMPLES. V.

1. A body moving with uniform acceleration describes 520 feet in the 7th second from rest; find the acceleration.

2. Starting from rest a body describes 330 feet in the 6th second of its motion; find the space described in the ninth second and in nine seconds.

3. A body whose initial velocity is zero has an acceleration of 32 ft.-sec. units; compare the distances passed over in the 6th and 12th seconds.

4. A body moving with uniform acceleration describes in the seventh and twelfth seconds after starting 23 and 33 feet respectively; find its initial velocity, its acceleration.

5. A body moving with uniform acceleration describes in the last second of its motion $\frac{5}{9}$ of the whole distance. It started from rest and described 6 inches in the first second; find how long it was in motion and the distance it described.

23. Falling bodies.

Experiments show that if a body fall it will move with an acceleration which is always the same in the same latitude and which is due to the attraction of the Earth. It is called the "acceleration of gravity."

Its measure is denoted by the symbol "g." When a foot and a second are the units the value of g is 32·2 approximately.

32·2 feet are equal to $\frac{32{\cdot}2 \times 12}{{\cdot}3937}$ centimetres or 981 centimetres, thus when a centimetre is the unit of length g is 981.

Thus from what has gone before we see that, taking g as 32,

velocity gained by a falling body in t seconds $= 32t$ ft. secs.,
distance fallen through $= 16t^2$ feet,

velocity gained in falling through a height $h = 8\sqrt{h}$ ft. secs.,

time occupied in falling through a height $h = \frac{1}{4}\sqrt{h}$ secs.

24. Motion under gravity.

The acceleration, due to gravity, of a body projected vertically upwards is *opposite* in direction to the velocity of projection, and is therefore denoted by $-g$.

In the formulae already proved put a equal to $-g$, we thus get

$$s = ut - \tfrac{1}{2}gt^2,$$

$$v = u - gt,$$

$$v^2 = u^2 - 2gs.$$

When the body is projected *downwards* we have the equations

$$s = ut + \tfrac{1}{2}gt^2,$$

$$v = u + gt,$$

$$v^2 = u^2 + 2gs.$$

If the body is let fall we have

$$s = \tfrac{1}{2}gt^2, \quad u = gt, \quad v^2 = 2gs.$$

25. Time to reach a given height.

If the given height be h, then writing h for s in the first equation of last Article, we have

$$h = ut - \tfrac{1}{2}gt^2.$$

This is a quadratic equation to find t, whose roots if real are both positive. If the roots are imaginary u is not great enough for the body to reach a height h.

The smaller root is the time at which the body is at the given height when *ascending*, the larger root gives the time at which it is at the same height when *descending*.

Ex. 1. A body is projected vertically upwards with a velocity of 80 ft. per sec.; when will it be at a height of 64 feet?

$$64 = 80t - 16t^2,$$

from which, $\quad t = 1$, or 4.

It is therefore at the given height *one* second or *four* seconds after the beginning of its motion.

Ex. 2. A body is projected vertically upwards with a velocity of 64 feet per second, how high will it rise in 2 seconds?

If x be the required height,

$$x = 64 \times 2 - 16 \times 4$$
$$= 64 \text{ feet.}$$

26. Velocity at a given height.

If h is a given height and u the velocity of projection, we have

$$v^2 = u^2 - 2gh, \text{ or } v = \pm \sqrt{u^2 - 2gh}.$$

Now u and h are the same whether the body is ascending or descending, hence the velocity is the same in magnitude but opposite in direction in these two cases.

Ex. 1. A body is projected with a velocity of 120 feet per second; what will be its velocity at the height of 200 feet?

If v be the required velocity

$$v^2 = (120)^2 - 64 \times 200,$$

or $$v = 40 \text{ feet per second.}$$

Ex. 2. If the upward velocity of projection be 60 feet per second, find when the velocity will be 20 feet per second.

If h be the height at which the velocity is 20,

$$(20)^2 = (60)^2 - 64 \times h;$$
$$\therefore h = \frac{(60)^2 - (20)^2}{64} = 50 \text{ feet.}$$

The times at which the body is at this height are found from the equation

$$50 = 60t - 16t^2,$$

from which $$t = 1\tfrac{1}{4} \text{ or } 2\tfrac{1}{2} \text{ seconds.}$$

27. Greatest height of a projected body.

At the highest point the velocity is zero; therefore if h is the height of the highest point the body reaches,

$$0 = u^2 - 2gh, \text{ or } h = \frac{u^2}{2g}.$$

Hence the velocity with which a body must be projected so as to just reach a height h is $\sqrt{2gh}$.

Ex. 1. A body is projected vertically upwards with a velocity of 64 feet per second, how high will it rise before beginning to descend? If h is the required height,

$$h = \frac{64^2}{64} = 64 \text{ feet}.$$

Ex. 2. A ball is allowed to fall to the ground from a certain height, and at the same instant another ball is thrown upwards with just sufficient velocity to carry it to the height from which the other falls. Show where and when the two balls will pass each other.

Let the given height be h. The ball thrown upwards with a velocity just sufficient to carry it this height must be thrown with velocity $\sqrt{2gh}$. Let the time up to the instant of meeting be t seconds.

Let s_1 be the distance described by the falling ball and s_2 that described by the one thrown upwards during the time t,

then $$s_1 = 16t^2,$$
$$s_2 = \sqrt{2gh}\,t - 16t^2;$$

hence adding $$s_1 + s_2 = \sqrt{2gh}\,t.$$

Put $$h = s_1 + s_2,$$
$$t = \sqrt{\frac{h}{2g}},$$

and $$s_1 = \frac{h}{4}.$$

28. Time to reach the greatest height.

At the highest point the velocity is zero,

$$\therefore 0 = u - gt, \text{ or } t = \frac{u}{g}.$$

29. Time of flight.

When the body has returned to the starting point the total space described is zero, therefore if t be the whole time,

$$0 = ut - \tfrac{1}{2}gt^2.$$

The roots of this quadratic equation are 0 and $\frac{2u}{g}$.

The root 0 corresponds to the time of starting, the root $\frac{2u}{g}$ is = time of ascent + time of descent = whole time of flight.

But $\frac{u}{g}$ was found to be the time of ascent, hence $\frac{u}{g}$ is also the time of descent.

Ex 1 A body projected upward returns to the point of projection at the end of six seconds, find the height reached and the velocity of projection

Since the time of flight is 6 seconds,

$$6 = \frac{2u}{g},$$

hence taking g as 32, the velocity of projection is 96 feet per second

The greatest height $= \frac{u^2}{2g} = \frac{(96)^2}{64} = 144$ feet

Ex 2 A ball thrown vertically upwards rises 200 feet in 4 seconds, in what time will it return to the point of projection?

By Art 24 we find the initial velocity to be 114 feet per second. Therefore the time of flight is $\frac{114}{16}$ or $7\frac{1}{8}$ seconds

30 The velocity gained by a fall from rest through a height h is $8\sqrt{h}$

The velocity with which a body must be projected to just reach a height h is by Art 26 also $8\sqrt{h}$

Hence the velocity gained by a fall through any height from rest is = velocity with which a body must be projected to just reach that height

Ex 1 A stone is thrown vertically upwards with such a velocity as will just take it to the level of the top of a tower 100 feet high Two seconds afterwards another stone is thrown up from the same place with the same velocity Determine when and where the stones will meet

The velocity of projection of the first stone is 80 feet per sec

The time of reaching the top is $\frac{80}{32}$ or $2\frac{1}{2}$ seconds

It therefore reaches the top half a second after the projection of the second stone This latter will in half a second reach the height

$$80 \times \tfrac{1}{2} - 16\left(\tfrac{1}{2}\right)^2 \text{ feet, or } 36 \text{ feet}$$

Its velocity is now $80 - 32 \times \frac{1}{2}$, or 64 feet per second

It is therefore moving with this initial velocity at the instant when the first stone is beginning to fall

Thus the spaces described by the two stones together with the space already described by the second stone or 36 feet is the height of the tower

$$36 + 64t - 16t^2 + 16t^2 = 100,$$

from which t is found to be 1 second

The time from when the first stone was thrown up is $3\frac{1}{2}$ seconds

Ex. 2. A stone is thrown upwards with a velocity due to a fall through a height $\frac{h}{2}$, from a point at a height h above the ground. Find when it will strike the ground.

The velocity due to a fall through a height $\frac{h}{2}$ is $\sqrt{gh}$. Art. 23.

The time of flight is therefore $2\sqrt{\frac{h}{g}}$. Art. 28.

Time of falling through the height h with the initial velocity $\sqrt{gh}$ is given by

$$h = \sqrt{gh}t + \tfrac{1}{2}gt^2. \quad \text{Art. 23.}$$

$$\therefore t = -\sqrt{\frac{h}{g}} + \sqrt{\frac{3h}{g}}.$$

The whole time before it strikes the ground is therefore

$$2\sqrt{\frac{h}{g}} - \sqrt{\frac{h}{g}} + \sqrt{\frac{3h}{g}}, \text{ or } (1+\sqrt{3})\sqrt{\frac{h}{g}}.$$

EXAMPLES. VI.

[g is taken to be 32 unless it is otherwise stated.]

1. How far will a body fall from rest in four seconds? Find with what velocity a body must be thrown upwards to return to the hand in four seconds.

2. A stone after falling for one second strikes a pane of glass, in breaking through which it loses half its velocity. How far will it fall in the next second?

3. A body moving from rest with a uniform acceleration passes over 10 feet in the first two seconds after starting, how far will it be from the starting-point at the end of the third second?

4. Through what vertical distance must a heavy body fall from rest to acquire a velocity of 161 feet per second? If it continue falling for another second after having acquired the above velocity, through what distance will it fall in that second?

5. A stone projected vertically upwards has risen 120 feet in one second. What was its initial velocity of projection, and how far will it rise during the next second?

6. If a stone reaches the ground again 6 seconds after it has been projected upwards, what was its height above the ground at the end of the first second?

7. A heavy particle is dropped from a given point, and after it has fallen for one second another particle is dropped from the same point. What is the distance between the two particles when the first has been moving for 5 seconds?

8. A balloon has been ascending vertically at a uniform rate for 4·5 seconds and a stone let fall from it reaches the ground in 7 seconds. Find the velocity of the balloon, and its height when the stone is let fall.

[The stone when let go has the velocity of the balloon (upward). If this be u the distance through which the stone falls is

$$-u \times 7 + 16 \times 7^2.$$

But this is the height to which the balloon rose in 4·5 seconds, or $u \times 4{\cdot}5$. Hence

$$u \times 4{\cdot}5 = 16 \times 49 - 7u;$$

$$\therefore\ u = 68{\cdot}2 \text{ feet per second.}$$

And height of balloon $= 68{\cdot}2 \times 4{\cdot}5$ feet

$= 306{\cdot}9$ feet.]

9. From a balloon which is ascending with a velocity of 32 feet per second a stone is let drop which reaches the ground in 17 seconds. How high was the balloon when the stone was dropped?

10. A rifle-bullet is shot vertically downwards from a balloon at the rate of 400 feet per second. How many feet will it pass through in two seconds, and what will be its velocity at the end of that time?

11. A stone dropped into a well reaches the water with a velocity of 80 feet per second, and the sound of its striking the water is heard $2\frac{7}{12}$ seconds after it is let fall. Find from these data the velocity of sound in air.

12. A ball is let fall from a height of 256 feet, and at the same time a ball is thrown upwards to meet it: find this velocity of projection if the balls meet at a height of 112 feet.

13. The space passed over in any given time may be represented by an area. Explain clearly the meaning of this statement and under what conditions it is true. Show how to employ it to determine the space passed over by a body in 10 seconds after it starts from rest, and has its velocity increased by one foot per second at the beginning of each second.

14. Explain by reference to a diagram why a stone only falls 16 feet during the first second while yet the force of gravity generates in that time a velocity of 32 feet per second.

15. Find the distance traversed in 10 minutes by a train which has at first a velocity of 20 miles per hour, and which has its speed in that time diminished at a uniform rate to 5 miles per hour.

16. A train running 60 miles an hour is pulled up by its brakes in 900 yards. At what rate would it be running if it could be pulled up in 100 yards?

17. Prove that if a body is projected vertically upwards with the velocity of 64 feet per second and 3 seconds afterwards a second body is let fall from the point of projection, the first body will overtake the second body $1\frac{1}{2}$ seconds later, 36 feet below the point of projection.

18. Prove that space described from rest in the n^2+n+1th second is = the sum of the spaces described in the first n seconds and in the first $n+1$ seconds.

19. A body moving with uniform acceleration f passes over a space s in a time T, show that its velocity at the beginning of the time was

$$\frac{s}{T}-f\frac{T}{2}.$$

20. The space described by a body in the 5th second of its fall from rest was to the space described in the last second but three as 9 : 11. For how many seconds did the body fall?

21. Supposing the moon to be a sphere of 2000 miles in diameter which rotates on its axis in 27 days, and that the circumference of a circle is to its diameter as 22 : 7, compare the velocity of a particle in the moon's equator with that of a railway train which travels 57 miles in an hour and a half.

22. At the ends of successive seconds the velocity of a body moving in a straight line has the following values: 4, $4\frac{1}{2}$, 5, 4, $2\frac{1}{2}$, 1, 0, -1, -1, 0. Draw the velocity-diagram, supposing the motion continuous, and estimate the space passed over.

23. A body is moving in a straight line; its distances from the start at the end of successive seconds are 0, 5·4, 10·5, 12·0, 11·5, 8·1, 0. By means of a diagram estimate the velocities at these points, and by charting the velocities determine the greatest, least, and mean acceleration throughout the motion.

24. Draw the velocity-time curve for a body thrown vertically upwards with a velocity of 10 ft.-sec. Use it to find (a) the time when the body is moving downwards with a velocity of 1 ft.-sec., (b) the distance of the particle from the point of projection at that time.

CHAPTER II.

THE LAWS OF MOTION.

31. Force and Mass.

Hitherto we have been dealing with the motion of a body without considering the means by which the motion has been produced, and the only quantities we have had to measure have been lengths and times; but when we consider the causes of the motion of bodies we have new quantities to define and measure. The science of Dynamics, then, treats of the forces acting on bodies and the motion produced by these forces; if the forces are in equilibrium and the bodies are at rest, the problem is referred to as one in Statics. As a result of common experience we know that, speaking generally, force or effort must be applied in order to produce motion in a given body. Again, we may have two bodies of the same size struck by blows of equal strength, but such that different velocities are produced in them; we recognize, then, that there is some property of the body which plays an important part in determining the effect of a given force. This property is that of Inertia, and its measure is called the Mass of the body. We must now obtain accurate definitions of these two quantities Force and Mass; the science of dynamics will, then, consist in the formulation and application of laws connecting the forces in operation with the masses and motions of the bodies acted upon. A set of propositions containing definitions of force and mass was first given by Sir Isaac Newton (1642—1727); these laws have been used as the basis of theoretical mechanics, and results calculated by them have been found to be consistent with observation and experiment. We shall state and discuss these laws.

32. First law of motion.

Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it is compelled to change that state by impressed force.

The first part of this law is a matter of common observation, namely, that force is required to produce motion in a body at rest. We can also recognize the grounds for the second part in ordinary experience. For instance, if a stone is projected along a surface of ice, it is brought to rest at last by the slight resistances due to the air and the friction between the stone and the ice; but if these are diminished more and more the stone goes continually further before being brought to rest. Hence we say that if a body is once set in motion with a given velocity and is afterwards not acted upon by any forces it will continue to move with that velocity uniformly in a straight line.

Now if a body is either at rest or moving with a velocity of unchanging amount and direction, it has in either case no acceleration. Thus the first law tells us to recognize force by acceleration. If we have a body whose velocity is changing, either in magnitude or direction, we know that some force is acting upon it; and conversely if force acts upon a body we look for its effect in acceleration. The second law must show a definite relation between these two quantities.

33. Second law of motion.

Change of momentum is directly proportional to the impressed force and to the time during which it acts, and is in the direction of the impressed force.

We have first to define the meaning of the term Momentum. It is the product of mass into velocity; *e.g.* if a body whose mass is m units is moving at any time with a velocity of v units, it is said to have mv units of momentum.

Now suppose a body of mass m to be initially at rest and to be acted on by a constant force F for a time t, and let v be the velocity of the body at the end of the time t; then the second law is one of direct proportion between the product Ft and the product mv, and may be written

$$Ft \propto mv,$$

or
$$F \propto \frac{mv}{t} \quad \text{.........................(1).}$$

But, by hypothesis, F and m are constant; hence v/t is the constant rate of change of the velocity during the time t, or, in other words, the *acceleration*. Hence the definite relation which the second law gives may be stated thus: the force acting on a particle is directly proportional to the product of the mass and the acceleration; hence we may write the law as

$$F = Cma \quad \text{.......................(2),}$$

where C is a numerical constant which is the same for all particles.

34. Measurement of Mass.

We can now specify accurately the measure of the mass of a given body. For we can compare the masses of any two given bodies by acting on them by equal forces (applied, for instance, by equal springs kept equally extended); then, if we measure the accelerations produced in the two bodies, it follows from the second law—since the forces F are the same—that the two masses are inversely proportional to the two accelerations. To obtain a measure of any mass we choose, then, some unit mass and suppose the given mass compared with this standard unit. There are two standard units of mass: one is the mass of a certain piece of platinum kept in London, and called the British Imperial Pound; the other is the mass of a cylinder of platiniridium kept at Sèvres, and called the Kilogramme. In the latter case masses are usually measured as a certain number of grammes, a gramme being the $\frac{1}{1000}$th part of the standard mass.

But the second law justifies a more practical method of comparing the masses of two bodies. For we know that if the only force acting on a body is its weight W, then it falls vertically with a constant acceleration g independent of the mass, constant for all masses at a given place. Then W is directly proportional to the product mg, and g is constant; hence W is directly proportional to m, and conversely. Thus to compare two masses we need only compare their weights at the same place; the masses are in the same proportion.

35. Units of Force.

We have to specify now the units in which force is measured.

Returning to equation (2), it will clearly be convenient in theoretical work to make the constant C equal to 1; this means that when m and a are each 1, the measure of F must also be 1.

Thus, if we write

$$F = ma \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

the unit force is that force which, acting on unit mass, produces in it unit acceleration. F is then said to be measured in absolute units of force.

If the unit mass is one gram, and the unit acceleration is one centimetre per second per second, the unit force is called the *Dyne*; this is the absolute unit of force which is used in scientific work.

If the unit mass is one pound, and the unit acceleration is one foot per second per second, the unit force is called the *Poundal.*

Since W equals mg absolute units of force, we see that the gram weight is g Dynes, or about 981 Dynes; while the lb. weight is g Poundals, or approximately 32 Poundals.

Another unit of force is used in England in practice, as for example in Engineering calculations; this unit is the *Pound Weight,* or the force with which the Earth attracts a lb. mass at its surface.

Now the lb. weight depends upon the value of g, the acceleration due to gravity, and we know that g varies slightly for different parts of the Earth's surface. Consequently, to make the lb. weight a definite unit of force, we define it to be that force which acting on the mass of 1 lb. produces in it an acceleration of 32·2 feet per second per second, that is, in the formula (2) we must have

$$F = 1, \text{ when } m = 1, \text{ and } a = 32{\cdot}2.$$

Hence $$C = \frac{1}{32{\cdot}2}.$$

Then, if we use the equation

$$F = \frac{ma}{32 \cdot 2} \text{ ....(4),}$$

we express m in lbs. mass, a in ft.-sec. units, and F is given in lbs. weight.

We have chosen the number 32·2, because the acceleration due to gravity at London has very nearly this value; thus the unit force just defined is practically the weight at London of the standard lb. mass.

36. Another way of writing the second law is worth notice in regard to the question of units. Suppose we observe a certain body moving in a straight line with acceleration a, and we require to estimate the force which must be acting on this body to produce in it the observed acceleration. Now we know another force, namely the weight W of the body, which acting on it would produce a known acceleration, namely g. But the second law states that the ratio of two forces acting on the same body is equal to the ratio of the accelerations produced by them; thus we have the simple proportion

$$\frac{F}{W} = \frac{a}{g}.$$

Hence
$$F = \frac{a}{g} W \text{ ....(5).}$$

In these equations a and g must be measured with the same units of length and time; then $\frac{a}{g}$ is a number, the ratio of two quantities of the same kind. Consequently F and W are also in like units; hence in whatever units we measure the weight W of the given body, F is given as a certain fraction or multiple of W measured in the same units.

36 *a*. If 1 foot, 1 second, and 1 lb. wt. are the units of length, time and force respectively, the equation $F = ma$ gives F in lbs. wt., provided the unit of mass is 32·2 pounds. For we have here $m = (\text{mass in pounds}) \div 32 \cdot 2$; hence from (4) F is given in lbs. wt.

If we write (5) in the form

$$F = \frac{W}{g} a \text{ ....(6),}$$

and compare (6) with (3), we see that W/g has taken the place of the mass m; then if (6) is to be the same equation as (3), we have

$$m=\frac{W}{g} \quad \text{.............................(7)}.$$

If the *unit mass* is one pound and g is 32·2 in foot-sec. units, $W=32{\cdot}2$ units of force when $m=1$; or the weight of 1 pound is 32·2 poundals. Similarly, if the unit mass is one gram and g is 981 in cm.-sec. units, the weight of one gram is 981 dynes. On the other hand, if the *unit force* is one lb. weight, then with $W=1$ and $g=32{\cdot}2$ we have from (7), $m=\frac{1}{32{\cdot}2}$. Hence the mass of 1 lb. wt. is $\frac{1}{32{\cdot}2}$ of a unit of mass; or the unit mass is 32·2 pounds.

37. Third law of motion.

To every action there is an equal and opposite reaction.

We have seen by the first law that we recognize the action of an unbalanced force by the production of acceleration in the body acted upon; then in the second law we found a definite relation between these two quantities. Now the third law states that all forces with which we have to deal occur in pairs. Thus every action between two bodies is of the nature of a pull or a push, a tension or a pressure; whether viewed as acting on one body or on the other it is in either case a given force in a certain straight line, but is in opposite directions in the two cases. If we have two bodies A and B acting on each other, for example by being pressed together or by means of the tension of a string, then the action of B on A is a force T in a certain line, and the action of A on B is an equal force T in the same line but in the opposite direction. These actions, if A and B were operated on by no other forces than their mutual action, would produce opposite accelerations in the two bodies along the line of action and inversely proportional to the two masses; but in general both A and B will be acted on by other external forces, so that the total effect may be quite different. Of all the forces which may act upon the two bodies A and B, their mutual action consists of pairs of equal and opposite forces.

38. Illustration of the laws of motion.

When two bodies are in motion connected by an inelastic string which remains stretched, the tension of the string is constant along its length provided the mass of the string is

so small that it may be neglected. Any straight portion of the string is acted on by three forces: the tensions at the two ends and the weight of the portion; and these three must be equivalent to a force given by the product of the mass and acceleration of the portion of string. Neglecting then the mass and weight of the string, we see that the tensions at the two ends must be equal and opposite forces. It can also be shown that the tension of a string is unaltered by passing over a perfectly smooth surface with which it is in contact.

Let two masses m and m' be connected by a light inextensible string passing over a fixed smooth peg. Let T be the tension of the string in absolute units of force, and suppose m greater than m'; then since the string remains stretched and of constant length, the velocity of m downwards at any instant equals the velocity of m' upwards at the same instant and consequently the accelerations of the two masses are numerically equal. Let a be the numerical value of the common acceleration at any time during the motion.

The mass m' if acted on only by its weight, equal to $m'g$ absolute units, would fall downwards with acceleration g; but it is made to move upwards with acceleration a. Hence the tension T of the string is certainly greater than $m'g$. Again, if the mass m were hanging at rest from the string, the tension would be mg; but the mass m is moving downwards, hence T is clearly less than mg.

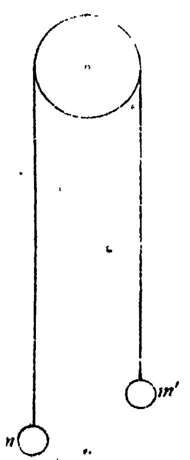

FIG. 6.

And similarly we see that a must be less than g.

But by the second law of motion we can write down accurately the relations that hold for each body separately, and find the actual values of T and a, knowing the values of m, m', and g.

The actual forces acting on m are mg vertically downwards and T vertically upwards. But the particle is a mass m moving with vertical

acceleration a downwards; hence the total force acting on it must be ma absolute units in the direction of a. Thus we have the two diagrams in fig. 7 representing two equivalent sets of forces.

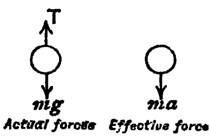

FIG. 7.

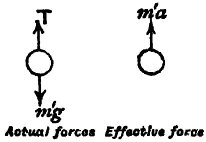

FIG. 8.

Hence we write down this equivalence as

$$mg - T = ma \quad \text{.....................(1).}$$

Similarly for the mass m' we have the diagrams of equivalent systems in fig. 8.

Hence $$T - m'g = m'a \quad \text{.....................(2).}$$

We have now two equations involving the two unknown quantities T and a. Solving these algebraical equations, we obtain

$$T = \frac{2mm'}{m + m'} g \quad \text{.....................(3),}$$

$$a = \frac{m - m'}{m + m'} g \quad \text{.....................(4).}$$

From these actual values we see that, as we expected, a is a certain fraction of g, while T lies between mg and $m'g$.

If lb.-foot-second units have been used then T is given by (3) in poundals; while if m, m', and g are in C.G.S. units then T is given in dynes.

We can also write down the equations in terms of the weights w and w' of the two bodies in the manner of Art. 36.

The equations corresponding to (1) and (2) are evidently

$$\frac{w - T}{w} = \frac{a}{g},$$

$$\frac{T - w'}{w'} = \frac{a}{g}.$$

Hence $$T = \frac{2ww'}{w + w'} \quad\text{.......................(5)},$$

$$a = \frac{w - w'}{w + w'} g \quad\text{.......................(6)}.$$

In these equations w and w' are expressed in terms of the same unit, and T is given also in terms of this unit. If w, w' are measured in absolute units, these expressions of course reduce to the previous results.

39. If we wish to solve any dynamical problem by direct use of Newton's laws we proceed as in the above example. We consider each mass of the system separately; we draw a diagram representing all the actual forces acting on the part and express that the combined effect of these is equivalent to a single force (mass) × (acceleration) in the direction of the acceleration; in general the actual forces on each mass will act in different directions, so that we shall have to see later how to combine together such a system. We shall then have obtained the dynamical equations of the system. There are also in addition geometrical equations representing the way in which the parts of the system are connected; for example, in the previous Article the geometrical relation is the constancy of the length of the string, and we use this fact in writing the accelerations of the two masses as numerically equal.

Ex. 1. A body weighing 4 lbs. starts from rest and is acted on by a force equal to the weight of $\frac{1}{4}$ lb. for 8 seconds, find the distance described.

The body has the mass of 4 lbs., ∴ $m = 4$.

The force equals $\frac{1}{4}$ lb. wt., or 8 poundals, ∴ $F = 8$.

Hence the equation $F = Ma$ gives

$$a = \frac{F}{M} = \frac{8}{4} = 2.$$

Also $$s = \tfrac{1}{2}at^2 = \tfrac{1}{2} \,.\, 2 \,.\, 8^2 = 64.$$

The space described equals 64 feet.

Ex. 2 A body whose weight is 1 cwt. is acted on by a force which produces in it an acceleration of 36 ft.-secs. per hour, find the magnitude of the force.

An acceleration of 36 ft.-secs. per hour is an acceleration of

$$\frac{12 \times 3}{60 \times 60} \text{ foot-secs. per second,}$$

or $\frac{1}{100}$ ft.-secs. per sec.; hence $a = \frac{1}{100}$.

Thus the measure of the force $= 112 \times \frac{1}{100} = 1 \cdot 12$. The force is 1·12 poundals.

Ex. 3. A force equal to the weight of 3 grammes acts on a body for 10 seconds, and causes it to describe 10 centimetres in that time, find the mass of the body.

If a is the acceleration, then since 10 cms. are described in 10 seconds we have

$$10 = \tfrac{1}{2}a(10)^2, \text{ or } a = \tfrac{1}{5} \textit{ in C.G.S. units.}$$

The equation $F = ma$ gives us that the force equals $\frac{m}{5}$ Dynes, but it is also equal to the weight of 3 grammes or 3×981 Dynes,

$$\therefore \frac{m}{5} = 3 \times 981,$$

$$m = 3 \times 5 \times 981$$

$$= 14715, \text{ or the mass is 14715 grammes.}$$

Ex. 4. A mass of 10 lbs. falls from rest through a distance of 10 feet and is then brought to rest by penetrating mud to the depth of 2 feet: find the resistance of mud (supposed uniform).

The velocity acquired in falling through 10 feet is $8\sqrt{10}$, Art. 23.

Also if F measures the force of resistance of the mud and a the resulting acceleration, we have $F - 10g = 10a$.

The body enters the mud with velocity $8\sqrt{10}$ which is reduced to zero in passing through 2 feet, hence since $v^2 = 2as$, Art. 20,

$$640 = 2a \,.\, 2, \text{ or } a = 160 \text{ ft./(sec.)}^2.$$

Hence $\quad F - 10g = 10a = 1600$ poundals,

$$\therefore F = 60 \text{ lbs. weight, nearly.}$$

EXAMPLES. VII.

1. Find the force which acting on a cwt. for 1 second gives it a velocity of 5 yards per minute.

2. A mass of 10 lbs. is placed on a smooth horizontal table and is acted on by a force equal to the weight of 10 lbs., find the distance it will describe in 1 second.

3. A mass of 400 tons is acted on by a force of 112000 poundals; how long will it take to acquire a velocity of 20 miles an hour?

4. In what distance will a force equal to the weight of 1 ounce be able to stop a mass of 40 lbs. which at the time the force begins to act has a velocity of 60 feet per second?

5. A bullet moving at the rate of 200 feet per second is fired into a thick target, which it penetrates to the extent of 6 inches; if fired into a target 3 inches thick with equal velocity with what velocity would it emerge, supposing the resistance the same in both cases?

6. A mass of 10 lbs. falls 100 feet and is then brought to rest by penetrating 1 foot into sand. Find the resistance of the sand.

7. If a force equal to the weight of 10 lbs. act upon a mass of 10 lbs. for 10 seconds what will be the momentum acquired?

8. A certain force acting on a mass of 10 lbs. for 5 seconds produces in it a velocity of 100 feet per second. Find the force and the acceleration it would produce in the mass of a ton.

9. A train of 200 tons weight is urged forward with a force equal to the weight of 1 ton, while it is retarded by a force equal to the weight of 10 lbs. per ton. What is its acceleration, and in what time will it acquire a velocity of 10 miles an hour?

[If F is the impelling and F' the retarding force, $F-F'=ma$.]

10. While a train travels half a mile on a level line its speed increases uniformly from 15 miles an hour to 30 miles. Find the ratio of the pull of the engine to the weight of the train.

11. A body resting on a smooth horizontal table is acted on by a horizontal force equal to the weight of 2 ounces, and moves on the table over a distance of 10 feet in 5 seconds starting from rest. What is its mass?

12. A force equal to the weight of a gramme acts on a body whose mass is 27 grammes for one second. Find the velocity of the body and the space passed over.

40. Atwood's Machine.

This machine is used to verify the laws of motion and also to determine the value of g. In an actual machine there are various mechanical refinements introduced to ensure accuracy of working and to diminish friction, but a diagrammatic sketch is sufficient to illustrate the theory of the machine.

A light string passes over a pulley and carries at its ends two equal masses each of weight P. On a graduated pillar AB a platform D and a ring E can slide up and down and can be fixed in any required positions. One of the weights P can pass through the ring, and there is also a bar of weight Q which is too long to pass through E.

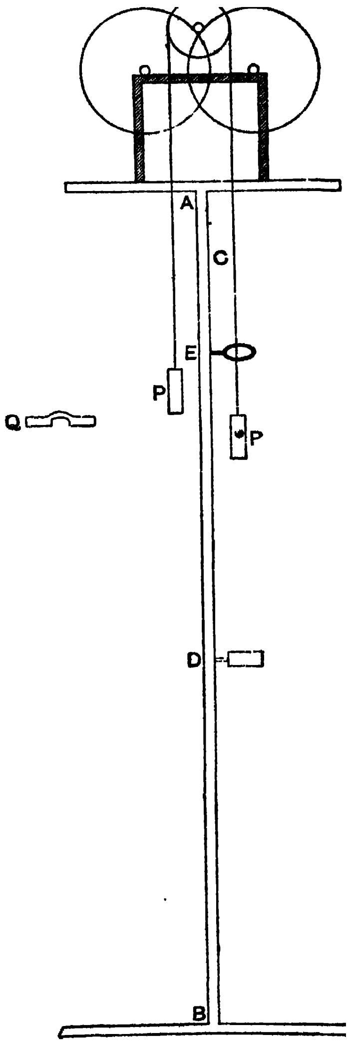

FIG. 9.

The bar Q is placed upon P at some point C, the parts being held at rest and then allowed to move. On the right we have now a weight $P+Q$ and on the left a weight P; consequently the weight $P+Q$ descends with acceleration

$$\frac{Q}{2P+Q}g.$$

Let h be the measured distance CE, then the velocity acquired when $P+Q$ reaches E is equal to

$$\sqrt{\frac{2Qgh}{2P+Q}}.$$

Q is now caught off by the ring, so that after this instant there are equal weights P at both ends of the string. Hence after this instant the system moves with constant velocity equal to its velocity at E given above. Then, if we observe

the time t taken by P to move from the ring E to the platform D, and measure the distance k from E to D, we have

$$k = t\sqrt{\frac{2Qgh}{2P+Q}},$$

or
$$k^2 = \frac{2Qght^2}{2P+Q}.$$

In this equation all the quantities have been measured except g; hence we can determine the value of g.

The first law can be verified by moving the platform D and testing whether the system does move with constant velocity in the second part of the motion when there is no unbalanced force acting on the system as a whole.

The second law could be tested by varying the weight of the rider Q, adjusting the ring E so that the same *time* is occupied from C to E in each case, and then find whether the subsequent uniform velocity is exactly proportional to the weight Q.

If these two laws are verified by the machine, then the third law is also; for it has been assumed in supposing the string to act upon the two masses with equal and opposite actions.

It must be noticed that in an actual machine, in spite of precautions, there will be some slight friction. We have also neglected the mass of the pulley which is set rotating by the motion; a more complete theory for an actual machine would take account of the rotation of the pulley as well as of the motion of the weights P and Q.

Ex. 1. The two ends A and B of a string passing over the pulley of an Atwood's machine are loaded as follows: A with $16\frac{1}{2}$ ounces, B with $15\frac{1}{2}$ ounces. Find the tension.

The forces acting on A are the tension T and the weight of $16\frac{1}{2}$ ounces, or 33 poundals, its mass is $\frac{33}{32}$ pounds, hence

$$\text{acceleration of } A = \frac{33 - T}{\frac{33}{32}},$$

similarly
$$\text{acceleration of } B = \frac{T-31}{\frac{31}{32}},$$

$$\therefore \quad \frac{33-T}{\frac{33}{32}} = \frac{T-31}{\frac{31}{32}},$$

or
$$T = \frac{33 \times 31}{32} \text{ poundals} = 16 \text{ oz. wt., nearly.}$$

Ex. 2. In Atwood's machine one of the two weights is heavier than the other by half an ounce. What must be the weight of each in order that the heavier one may fall through one foot in the first second?

Since the weight falls through one foot in the first second its acceleration is 2. Also if w_1 ounces be the weight of the lighter weight, that of the heavier is w_1+1.

Hence by Art. 38,

$$2 = \frac{w_1 + \frac{1}{2} - w_1}{w_1 + \frac{1}{2} + w_1} g,$$

or
$$2 = \frac{1}{2w_1 + \frac{1}{2}} \, 32,$$

$$w_1 = 3\tfrac{3}{4} \text{ oz.}$$

Thus the lighter weight is $3\frac{3}{4}$ oz.

...........heavier........... $4\frac{1}{4}$ ....

Ex. 3. A weight Q on a smooth table is connected by a string with a weight P which hangs vertically. Find the acceleration of P and Q.

If f is the acceleration of either

$$\frac{P}{g} f = P - T,$$

$$\frac{Q}{g} f = T.$$

Hence by addition $(P+Q)\dfrac{f}{g} = P,$

or
$$f = \frac{Pg}{P+Q}.$$

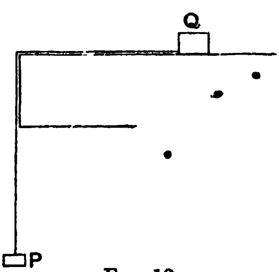

FIG. 10.

EXAMPLES. VIII.

1. Two masses of 48 and 50 grammes respectively, are attached to the string of an Atwood's machine, and starting from rest the greater mass falls through 10 centimetres in one second. Find the acceleration due to gravity.

2 A mass of 7 lbs is attached to one end of a string passing over a pulley, and two masses of 3 lbs and 6 lbs to the other end After 4 seconds the smallest mass is detached How much farther will the 6 lbs fall, and when will it be brought to rest?

3 Masses of 3 lbs and 5 lbs respectively are connected by a string passing over a pulley, after one second the string breaks, for how long and how far will the 3 lbs ascend?

4 Two weights of 7 oz and 9 oz are connected by a string 40 feet long which is hung over a smooth pulley so that the 7 oz weight touches the ground When the system has been in motion for two seconds the string is cut, and both weights reach the ground at the same time Find the height of the pulley from the ground

41 Further Illustrations of the Laws of Motion.

(i) A shot of mass m leaves the barrel of a gun of mass M with a horizontal velocity v, while the velocity of recoil of the gun is V, the gun being horizontal During the time the shot is in the barrel of the gun the vertical forces acting on the system are the two weights and the upward reaction of the horizontal plane on which the gun stands; but there is no vertical motion, hence the pressure on the plane is equal to the sum of the two weights

Again the explosive force of the powder acts equally forward on the shot and backward on the gun, and for the same time Hence the forward momentum of the shot is equal to the backward momentum of the gun, that is

$$mv = MV$$

or

$$v = \frac{m}{M} V.$$

If we regard the shot and gun as one system during the time in question, then the only horizontal force acting is an internal force which acts equally and oppositely on the two parts of the system, hence, on the whole, the total horizontal momentum is unaltered, or

$$MV - mv = 0.$$

(ii) When a heavy body rests on a horizontal platform the pressure of the platform on the body is equal and opposite to the pressure of the body on the platform. If

the platform is at rest the pressure is equal to the weight of the body. If the platform is made to move vertically downwards with acceleration a, the pressure will clearly be less than the full weight w of the body; let it be P, expressed in the same units as w. Consider now the body separately as under the action of two forces, w vertically downwards and P vertically upwards; then its acceleration downwards must be a certain fraction of g, namely

$$\frac{w-P}{w} g.$$

But its acceleration is made to be a downwards; hence

$$\frac{w-P}{w} g = a,$$

or

$$P = \frac{g-a}{g} w,$$

thus the pressure is a certain fraction of the full weight w.

Again, suppose the platform to be made to ascend with acceleration a. The pressure P exerted on the body by the platform must be greater than w; and by the same reasoning as before we have

$$\frac{P-w}{w} g = a,$$

or

$$P = \frac{g+a}{g} w,$$

the pressure being a certain multiple of the weight w.

Ex. 1. A body whose weight is 10 lbs. is placed on a horizontal plane moving vertically upward; if the pressure of the body on the plane is equal to the weight of 16 lbs. find the acceleration.

Here $$a = \frac{16-10}{10} g = \frac{6}{10}(32) = 19{\cdot}2.$$

Ex. 2. A balloon ascends vertically with uniform acceleration, so that a weight of 2 pounds exerts a pressure on the bottom of the car equal to the weight of $32\frac{1}{2}$ oz.; find the height the balloon will reach in one minute.

The force on the body is $65-64$ poundals, or one poundal. The mass of the body is 2, the acceleration is therefore $\frac{1}{2}$. Hence the height reached in one minute is

$$\tfrac{1}{2} \cdot \tfrac{1}{2}(60)^2 \text{ feet, or } 300 \text{ yards.}$$

EXAMPLES. IX.

1. A body whose weight is 112 lbs. is placed on a lift which moves with a uniform acceleration of 12 ft.-sec. units. Find the pressure on the floor when the lift is descending.

2. The pressure on the bottom of a bucket which is being drawn up the shaft of a mine is equal to the weight of 133 lbs.; if the contents of the bucket weigh 1 cwt. what is the acceleration?

3. A body whose weight is 1 stone is placed on a lift moving with uniform acceleration of 12 ft.-sec. units; find the pressure on the floor of the lift when it is (i) descending, (ii) ascending.

4. A mass of 40 lbs. rests on a horizontal table which is made to ascend, (i) with a constant velocity of 2 feet per second, (ii) with a constant acceleration of 8 ft.-sec. units; find in each case the pressure on the plane.

5. A man suddenly jumps off a table with a 20 lb. weight in his hand, what is the pressure of the weight on his hand while he is in the air?

6. A balloon ascends with constant acceleration, so that a mass of 56 lbs. exerts a pressure of 84 lbs. on the bottom of the car. When will it be 200 feet high and what will be its velocity at that time?

7. A cord passing over a smooth pulley supports two scale-pans, the weight of each being 3 ounces. If weights of 4 and 6 ounces be placed in the scale-pans find the acceleration and the tension of the string, also the pressure between the masses and the scale-pans.

8. Of two forces one acts on a mass of 5 lbs. and produces in it a velocity of 5 feet per second in $\frac{1}{11}$ of a second, the other acts on a mass of 625 lbs. and produces in it a velocity of 18 miles per hour in one minute, compare the forces.

9. A ball whose mass is 3 lbs. is moving at the rate of 100 feet per second. What force expressed in lbs. weight will stop it (i) in 2 seconds, (ii) in 2 feet?

10. A cannon-ball weighing 600 lbs. and moving with a velocity of 1000 feet per second penetrates a target to a depth of 15 inches. Find the pressure on the target, supposing it to be uniform.

11. A shot of 1000 lbs. leaves a gun with a velocity of 1500 feet per second. How long must the shot have been under the action of the powder supposing the average pressure upon it to have been equal to the weight of 1200 tons?

12. A balloon is moving upwards with a speed which is increased at the rate of 4 feet per second in each second; find by how much the weight of a body of 10 lbs. as tested by a spring balance in the balloon would differ from its weight under ordinary circumstances.

13. In what time will a weight of 16 lbs. draw another of 12 lbs. over a fixed pulley through 32·2 feet, and what velocity will the weights have at the end of the time?

14. The two weights in an Atwood's machine are 240 grammes each, and an additional weight of 10 grammes being placed on one of the two it is observed to descend through 10 metres in 10 seconds, show that $g=980$ nearly.

15. A man of 12 stone weight and a sack of 10 stone weight are connected by a rope over a smooth pulley. The man pulls himself up by the rope and diminishes his downward acceleration by $\frac{1}{2}$, find the upward acceleration of the sack and show that the acceleration of the man relative to the rope is 3·2.

16. A rope hangs over a smooth pulley and a man of 12 stone lets himself down with acceleration f', while a man of $11\frac{1}{2}$ stone pulls himself up with acceleration f. Find f' in order that the rope may remain at rest.

17. Two weights of 5 lbs. and 7 lbs. respectively are fastened to the ends of a cord passing over a frictionless pulley supported by a hook. Show that when they are free to move the pull on the hook is $11\frac{2}{3}$ lbs. weight.

18. Two equal weights A and B connected by an inelastic thread 3 feet long are laid close together on a smooth horizontal table 3·5 feet from the nearest edge, and B is also connected by a stretched inelastic thread with an equal weight C hanging over the edge. Determine the velocities of the weights when A begins to move and also when B arrives at the edge of the table.

19. One end of a string is fixed, it then passes under a moveable pulley to which a weight W is attached. The string then passes over a fixed pulley and a (smaller) weight P is attached to the other end, all the 3 sections of the string are vertical. Prove that the acceleration of W is $\frac{W-2P}{W+4P}g$. The weights of the pulleys are neglected.

20. A man hangs by a rope passing over a fixed pulley to the other end of which a weight equal to that of the man is attached. Prove that he cannot raise himself up above the level of the weight as he climbs up the rope.

21. To one end of a string passing over a fixed pulley hangs a weight P, and to the other end a pulley over which passes a string at one end of which hangs a weight Q and at the other a weight R resting on a table. The system being allowed free motion find the pressure on the table supposing R so heavy that it is not raised from the table.

22. To one string of an Atwood's machine a mass of P lbs. is attached. To the other string a mass of Q lbs. is attached where $P > Q$. Above the mass of Q lbs. is placed a mass of $\frac{P^2 - Q^2}{Q}$ lbs. which can be detached in the ordinary manner during the motion. The system starts from rest and moves for t seconds, at the end of which time the mass of $\frac{P^2 - Q^2}{Q}$ lbs. is detached. Show that in t seconds more the system will be for an instant at rest, and that the motion will then be reversed in direction.

23. A train of mass 120 tons is travelling with uniform speed, the resistance due to friction, etc. being 14 lbs. weight per ton. If a portion of mass 20 tons is slipped, how much will the other portion have gained on it in 12 seconds, assuming the pull of the engine and the resistance per ton to be the same as before?

CHAPTER III.

COMPOSITION AND RESOLUTION OF VECTORS.

42. Resultant.

When two or more forces act on a particle the single force which would produce their combined effect is called their *resultant*. If the forces are all in the same straight line the resultant is obtained by adding or subtracting the components according to their directions; we have now to obtain a method of combining forces which act along different lines on a particle. We can do this by a fuller interpretation of Newton's second law of motion. For each separate force acting on the particle would by itself produce an acceleration in its own direction proportional to its strength; consequently the resultant acceleration is the combination of component accelerations in the directions of the component forces and proportional to them. Also the resultant force must be in the direction of the resultant acceleration and proportional to it. Therefore component forces acting on a particle are combined together by the same rules as for compounding accelerations, that is, by the Parallelogram Law to be proved shortly.

We can class together Forces, velocities, accelerations as examples of physical quantities which, to be known, must be given in direction as well as in magnitude. Such quantities are called *Vectors*; other examples we have had are Displacement and Momentum.

On the other hand physical quantities such as Mass, Volume, Temperature, which we know completely on specifying their magnitude, are called *Scalars*. Any two scalars of the same kind are "added" by arithmetical addition or subtraction. But two vectors of the same kind are always added by the parallelogram rule.

43. The Parallelogram Law.

(i) *Two Displacements.* Suppose a particle is to be given two displacements of specified magnitude and direction. Let O represent the initial position of the particle and draw OA, OB in the given directions representing to scale the corresponding displacements. Completing the parallelogram $OACB$ (fig. 11), it is obvious that C represents the final position in whatever order the component displacements are made. Hence the diagonal OC represents in magnitude and direction the single displacement which would effect the same change in position as the two given displacements combined, or OC represents the resultant displacement.

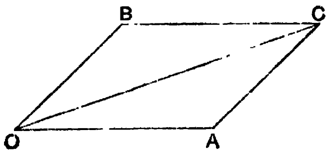

FIG. 11.

(ii) *Two Velocities.* Suppose now the two displacements represented by OA, OB to be carried out concurrently at a uniform rate during one second, so that at the end of the second the particle is at C. We may imagine this to be performed in the following manner. Let a line $O'A'$ move so that it is always parallel to OA with the point O' moving uniformly over the length OB in one second, while at the same time a point P moves uniformly along the line $O'A'$ from O' to A' (fig. 12). Then the motion of the point P represents the motion of a particle under the combined effect of two component velocities represented in magnitude and direction by OA and OB; let these component velocities be u and v units respectively.

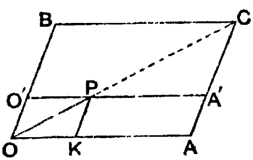

FIG. 12.

Let P be the position of the moving point after any fraction T of a second, and draw PK parallel to OB. Then since the velocities are uniform we have

$$OK = uT; \quad PK = vT.$$

Hence
$$\frac{PK}{OK} = \frac{v}{u} = \frac{CA}{OA}.$$

Therefore P lies on OC and hence at every intermediate

instant the particle is on the diagonal OC, and describes the diagonal OC uniformly in one second. Thus OC represents in magnitude and direction the resultant velocity equivalent to the two component velocities u and v.

(iii) *Two Accelerations.* Again, let the velocity of the particle be changed by an amount represented by OA, and after that by an amount represented by OB, in magnitude and direction. From the previous section it follows that the diagonal OC represents completely the final change in the velocity of the particle. Now suppose these two changes in velocity to be carried out concurrently, uniformly, and in one second. Then OA and OB represent completely two component accelerations, and we see that the diagonal OC gives the resultant acceleration in the same way.

(iv) *Two Forces.* We have seen that the separate forces would produce component accelerations in their own directions and proportional to them. Hence if OA and OB represent now the component forces in size and direction, OA and OB represent also the accelerations produced, of which OC represents the resultant, so that the resultant force is represented fully by OC.

A simple experiment can be made to verify directly the parallelogram of forces.

44. Verification of the Parallelogram of Forces.

Take three flexible strings and tie them together. Let two of them pass over smooth pegs at any distance apart, the third hanging freely. Attach weights P, Q, and R to their ends.

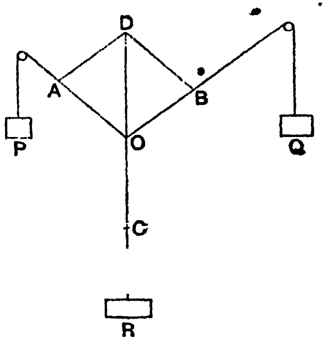

FIG. 13.

Let the system right itself; when it has settled, measure off on the strings lengths OA, OB and OC proportional to P, Q and R; then the tensions in the three strings, being equal to P, Q, and R are proportional to OA, OB and OC.

Complete the parallelogram

AOBD. It will be found by measurement that *OD* is equal in magnitude and opposite in direction to *OC*.

But the effect of the force represented by *OC* is equal and opposite to the joint effect of the forces represented by *OA* and *OB*, since the forces balance each other; hence *OD* represents the resultant of the forces represented by *OA* and *OB*.

45. Special cases of the Parallelogram Law.

The following simple cases are stated for two component velocities of a point; they hold equally for all vector quantities, for example, for two component forces acting on a particle.

(i) When the velocities are at right angles to each other.

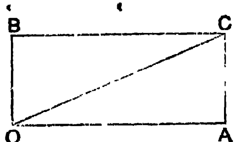

Fig. 14.

Let *OA* and *OB* represent the velocities u and v.

Then since $OC^2 = OA^2 + AC^2$,

$$(\text{resultant})^2 = u^2 + v^2,$$

or if w is the magnitude of the resultant,

$$w = \sqrt{u^2 + v^2}.$$

(ii) When the contained angle is 30°.

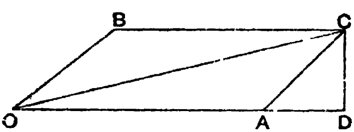

Fig. 15.

Draw *CD* perpendicular to *OA*. Then since *CAD* (or *BOA*), is 30°, *ACD* is 60°, and the triangle *CAD* is half an equilateral triangle;

$$\therefore \quad CD = \frac{AC}{2} = \frac{v}{2},$$

$$AD = \sqrt{AC^2 - \frac{AC^2}{4}} = \frac{\sqrt{3}}{2} AC = \frac{\sqrt{3}v}{2}.$$

But $$OC^2 = OD^2 + CD^2,$$

or $$OC^2 = (OA + AD)^2 + CD^2,$$

hence $$w^2 = \left(u + \frac{\sqrt{3}}{2} v\right)^2 + \frac{v^2}{4};$$

$$\therefore \quad w^2 = u^2 + v^2 + uv\sqrt{3}.$$

(iii) When AOB is 45°.

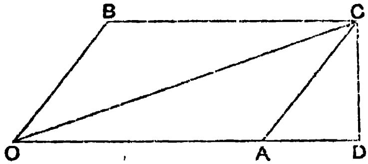

FIG. 16.

Since AOB is 45°, ACD is 45°, and $AD = CD$.

Hence $$AC^2 = 2AD^2, \text{ or } AD = \frac{AC}{\sqrt{2}}.$$

Thus $$OC^2 = \left(OA + \frac{AC}{\sqrt{2}}\right)^2 + \frac{AC^2}{2}.$$

Or, $$w^2 = \left(u + \frac{v}{\sqrt{2}}\right)^2 + \frac{v^2}{2}$$

$$= u^2 + v^2 + uv\sqrt{2}.$$

(iv) When AOB is 60°.

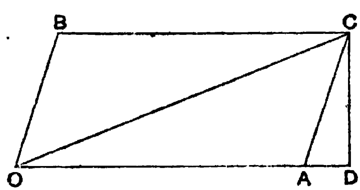

FIG. 17.

Here ACD is 30°; and ACD is half an equilateral triangle, hence as above

$$AD = \frac{AC}{2}, \quad CD = \frac{\sqrt{3}}{2} AC,$$

$$OC^2 = (OA + AD)^2 + CD^2,$$

$$w^2 = \left(u + \frac{v}{2}\right)^2 + \frac{3v^2}{4} = u^2 + v^2 + uv.$$

(v) When AOB is 120°.

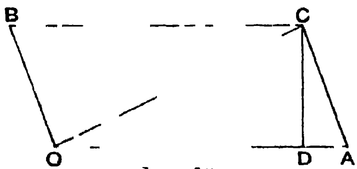

FIG. 18.

DAC is 60° and $OC^2 = OD^2 + CD^2 = (OA - AD)^2 + CD^2$

$$\therefore \quad w^2 = \left(u - \frac{v}{2}\right)^2 + \frac{3v^2}{4} = u^2 + v^2 - uv.$$

(vi) When AOB is 135°.

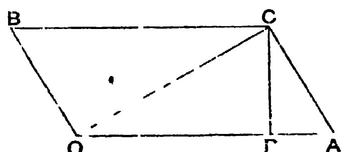

FIG. 19.

DAC is 45°, and $AD = DC = \dfrac{AC}{\sqrt{2}}$.

$$w^2 = \left(u - \frac{v}{\sqrt{2}}\right)^2 + \frac{v^2}{2} = u^2 + v^2 - uv\sqrt{2}.$$

(vii) When AOB is 150°.

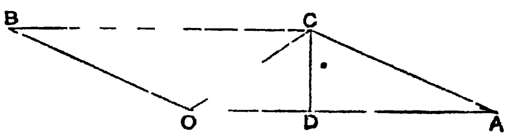

FIG. 20.

Here $DAC = 30°$, and

$$CD = \tfrac{1}{2}AC, \quad AD = \frac{\sqrt{3}}{2}AC.$$

Hence
$$w^2 = \left(u - \frac{\sqrt{3}}{2}v\right)^2 + \frac{v^2}{4} = u^2 + v^2 - \sqrt{3}uv.$$

45 *a*. All the preceding cases can be got from the relation

$$OC^2 = OA^2 + OB^2 + 2OA \cdot OB \cos\alpha,$$

where α is the angle between OA and OB.

This gives
$$w^2 = u^2 + v^2 + 2uv\cos\alpha.$$

EXAMPLES. X.

1. Find the resultant velocity in the following cases:

(i) $u = 1$, $v = 2$, including angle 30°.

(ii) $u = 3$, $v = 3$, ... 45°.

(iii) $u = 5$, $v = 11$, ... 60°.

(iv) $u = 1$, $v = 20$, ... 150°.

2. From the top of the interior of a railway carriage a stone is let fall. If the train is moving at the rate of 20 miles an hour, show that the velocity of the stone is $\frac{8}{3}\sqrt{130}$ feet per second when it has fallen one foot.

3. A balloon rising vertically with a velocity of 30 feet per second is also carried by the wind over a horizontal distance of 40 feet in a second. Find its total velocity.

4. Find the resultant of two velocities, of 10 feet and 20 feet per second respectively, inclined at an angle of 120°.

5. A ship sailing westwards with a velocity of 16 knots receives an additional velocity of 16 knots from a current so that its velocity is still 16 knots. What is the direction of the additional velocity?

6. A man walks in 12 seconds across the deck of a ship which is sailing due north at the rate of 4 miles an hour, and finds that he has moved in a direction 30° east of north. How wide is the deck and what is his actual velocity?

7. If two velocities of 9 feet and 7 feet per second respectively, possessed by a body, include an angle whose cosine is $\frac{1}{3}$, show that the resultant velocity is 12 feet per second.

8. Find the resultant of the following forces:

(i) forces of 5 and 11 lbs. acting at an angle of 30°;

(ii) .... 1 ... $\sqrt{2}$ .... 60°;

(iii) .... $\sqrt{3}$ ... $\sqrt{3}$ .... 60°;

(iv) .... 1 ... 4 ....150°.

9. Show that the resultant of two equal forces bisects the angle between their directions.

10. Two forces acting at right angles are to each other in the ratio of 1 to $\sqrt{7}$, and their resultant is 8 lbs.; find the forces.

11. Two forces acting at an angle of 120° have a resultant equal to $2\sqrt{3}$ lbs. weight; if one of the forces is 4 lbs. weight, find the other.

12. Find the magnitude of two forces such that if they act at right angles their resultant is $\sqrt{14}$ lbs. weight, while when they act at an angle of 120° their resultant is $\sqrt{13}$ lbs. weight.

13. A force equal to the weight of 15 lbs. acting vertically upwards is resolved into two forces, one being horizontal and equal to the weight of 10 lbs.; find the other force.

14. The magnitudes of two forces are as 4 : 5; the direction of their resultant is at right angles to the smaller force; find the ratio of the larger force to the resultant.

15. The resultant of two forces P and Q acting at a certain angle is R, the resultant of P and Q' acting at the same angle is R'; what would be the resultant of R and a force equal and opposite to R'?

16. In a triangle ABC, D, E and F are the middle points of the sides, show that forces acting at a point and represented by $2AD$, $2BE$ and $2CF$ are in equilibrium.

17. A vertical force of 10 lbs. is resolved into two equal components, one of them making an angle of 30 with the vertical; find the magnitude and direction of the other.

18. If the directions of two forces be inclined to one another at an angle of 135°, find the ratio of their magnitudes that their resultant may be equal to the smaller force.

19. Find the components in directions due E. and N.W. of a force equal to the weight of 12 lbs. acting N.E.

20. If a given force acting at a given point in a given direction be resolved into two equal forces, prove that the extremities of the lines representing the equal forces always lie on a fixed straight line.

21. The resultant of forces of 5 lbs. and 6 lbs. is 7 lbs.; find the cosine of the angle between the forces of 5 and 6 lbs.

22. The direction of a force of 10 lbs. weight makes an angle a with the horizon such that $\cos a = \frac{3}{5}$; find its horizontal and vertical components.

23. AB, AC represent forces of 33 and 25 lbs. respectively. If CD were drawn perpendicular to AB, AD would represent on the same scale 15 lbs., prove that the resultant of the forces in AB and AC is 52 lbs.

24. A force $8P$ is resolved into two forces, each of which is equal to $5P$, find the sine of the angle between the equal components.

46. The Triangle Rule.

In the figure (Art. 43) the two component vectors are represented by OA, OB; but AC is parallel and equal to OB, hence we have the method of combining vectors as a triangle rule: if two vectors acting at a point are represented in magnitude and direction by two sides of a triangle taken in order, the third side taken in the opposite way round the triangle will represent their resultant.

47. The Polygon of Vectors.

When there are more than two component vectors a continued application of the triangle rule leads to the final resultant. For example, consider a particle which possesses any number, say four, component velocities u_1, u_2, u_3, u_4. From a point A draw AB representing to scale and in direction the component velocity u_1; from the end point B draw BC representing u_2. Then draw CD to represent u_3,

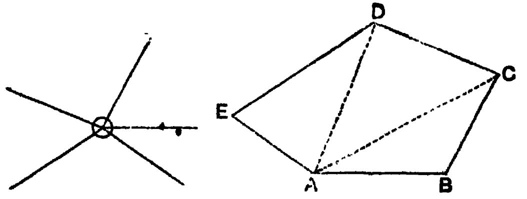

FIG. 21.

and finally DE representing u_4 in size and direction. Then by the triangle rule we have

Resultant of AB and BC is AC,
.... AC and CD is AD,
.... AD and DE is AE.

Hence the line AE required to close the polygon, drawn from the initial point A to the end point E, represents completely the resultant velocity. The method evidently holds for any number of components and for all other vectors, such as forces and accelerations.

48. Equilibrium.

If the end point E is found to coincide with A, the polygon is closed and the resultant is zero. For component velocities possessed by a particle, this means that the particle is at rest. For component forces acting on a particle, the forces are then in equilibrium; the following case is of special importance.

49. Triangle of Forces.

Given three forces acting at a point we proceed to draw the polygon of forces. If the end point coincides with the starting point the figure is a triangle ABC. Then the resultant of R and P is represented completely by AC.

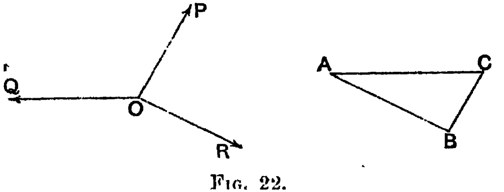

Fig. 22.

But Q is represented completely by CA. Hence the resultant of R, P and Q is zero and we have the following theorem:

If three forces acting at a point can be represented by the sides of a triangle taken in order, they are in equilibrium.

50. Converse of the Triangle of Forces.

The converse of the triangle of forces is also true, viz. that *if three forces acting at a point are in equilibrium, any triangle which has its sides parallel to the forces will have those sides also proportional to the forces.*

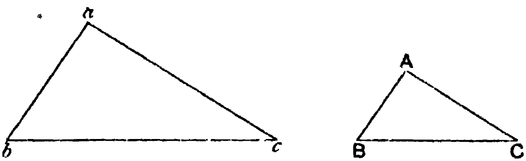

FIG. 23.

Let the sides of the triangle abc be parallel to the forces P, Q, R which are in equilibrium, it is required to prove they are also proportional to P, Q and R.

Let the sides BC and CA of the triangle ABC represent the forces P and Q in magnitude and direction. Then since forces represented by BC, CA and AB are in equilibrium by the Triangle of Forces, the side AB must represent R in magnitude and direction.

Hence $$BC : CA : AB = P : Q : R.$$

Also the sides of the triangle abc are parallel to those of ABC, therefore abc and ABC are similar triangles, and

$$BC : CA : AB = bc : ca : ab.$$

Therefore $$P : Q : R = bc : ca : ab.$$

51. Lami's Theorem.

When three forces are in equilibrium each force is proportional to the sine of the angle between the directions of the other two.

The forces P, Q and R are in equilibrium, and ABC is any triangle having its sides parallel to their directions (fig. 22). Then by the Converse of the Triangle of Forces

$$P : Q : R = BC : CA : AB.$$

The angles of the triangle ABC are supplementary to those between the directions of P, Q and R, and since

$$BC : CA : AB = \sin A : \sin B : \sin C,$$

we have $$P : Q : R = \sin A : \sin B : \sin C$$

or, $$P : Q : R = \sin QOR : \sin ROP : \sin POR.$$

Ex. 1. Three forces acting at a point are in equilibrium; if they are equal find the angle between their directions.

By the Converse of the Triangle of Forces, the forces can be represented by the sides of an equilateral triangle.

They therefore make angles of 120° with one another.

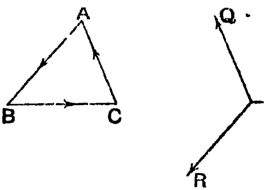

FIG. 24.

Ex. 2. A weight of 24 lbs. is suspended by two flexible strings one of which is horizontal, and the other is inclined at an angle of 30° to the vertical; what is the tension in each string?

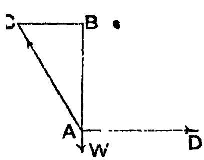

FIG. 25.

AD and AC being the strings, the sides of the triangle ACB are parallel to the forces which are the tensions of the strings and the suspended weight.

The angle CAB is 30°, the angle ABC is 90°, hence

$$BC = \tfrac{1}{2}AC, \quad AB = \sqrt{3}BC.$$

Therefore tension in AC : suspended weight

$$= AC : AB = \frac{2}{\sqrt{3}};$$

or, $$\text{tension in } AC = \frac{2}{\sqrt{3}} \times 24 \text{ lbs.} = 16\sqrt{3} \text{ lbs.}$$

And tension in AD : suspended weight $= BC : AB = \dfrac{1}{\sqrt{3}}$;

or, $$\text{tension in } AD = \frac{1}{\sqrt{3}} \times 24 \text{ lbs.} = 8\sqrt{3} \text{ lbs.}$$

Ex. 3. Three forces whose magnitudes are 3, 6 and 9 lbs., act at a point in the directions of the sides of an equilateral triangle taken in order, find their resultant.

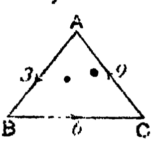

FIG. 26.

By the triangle of forces, the forces 3, 3, 3, in the assigned directions are in equilibrium, they may therefore be *removed.*

There are left forces of 3 and 6 lbs. acting at an angle of 120°.

Their resultant is therefore $3\sqrt{3}$ lbs.

EXAMPLES. XI.

1. Three forces acting at a point are in equilibrium; the greatest force is 5 lbs., the least 3 lbs., and the angle between two of the forces is a right angle. Find the other force.

2. Three forces represented by the numbers 1, 2, 3, act on a particle in directions parallel to the sides of an equilateral triangle taken in order; find their resultant.

3. A weight of 10 lbs. hangs fastened to the ends of two strings, the lengths of which are 3 and 4 feet, the other ends of the strings being attached to two points in a horizontal line distant 5 feet from each other; find the tension of each string.

4. Three forces cannot be in equilibrium if the sum of any two is less than the third.

5. At what angle must two equal forces act so that their resultant may equal each of them?

6. If three forces P, P, Q act at a point in directions such that each force is equally inclined to the directions of the other two, find their resultant.

7. A body is acted on by three forces, one of 2 lbs. due west, one of 4 lbs. north-east, and one of $2\sqrt{2}$ lbs. due south, find their resultant.

8. When two of three forces in equilibrium are given in magnitude, the third force increases as the angle between the first two forces diminishes.

9. Give a geometrical construction for resolving a force into two others inclined at a given angle, one of which is to be of given magnitude.

10. A weight of 10 lbs. is supported by two forces, one of which is horizontal and the other inclined at 30° to the horizon. Find the forces.

11. Forces of 5 lbs., 6 lbs. and 7 lbs. acting at a point are in equilibrium; find the cosines of the angles between them.

12. A machine of 5 tons weight is supported by two chains, one being inclined 20° and the other 73° to the horizontal; find the pulling forces in the chains.

51 *a*. Having *given* that the *direction* of the resultant of two forces is that of the diagonal of the parallelogram of which the lines representing the forces form two adjacent sides, we may prove that this

diagonal represents the resultant in *magnitude* also in the following manner.

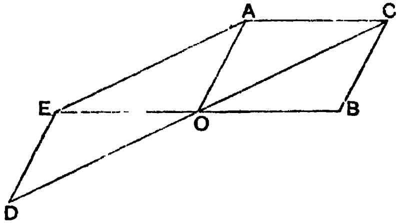

FIG. 27.

Let OA and OB represent two forces, then assuming that the direction of their resultant is that of OC we have to show that its magnitude is represented by OC. Produce CO to D so that OD represents the magnitude of the resultant. Complete the parallelogram $AODE$. Join OE.

Then the forces represented by OA, OB and OD are in equilibrium, hence OB is in the same line with the resultant of the forces represented by OA and OD; but we are given above that OE, the diagonal of the parallelogram $AODE$, is the direction of this resultant; hence B, O and E are in the same straight line.

Then since the figure $ACOE$ is a parallelogram, OC is equal to AE, and since the figure $AODE$ is a parallelogram, OD is equal to AE, therefore OD is equal to OC, which was to be proved.

Ex. Five equal forces act on a particle in directions parallel to five consecutive sides of a regular *hexagon* taken in order; find the magnitude and direction of the resultant.

By the Polygon of Forces the resultant is represented by the line closing the figure drawn *from* the starting point; hence it is equal to any one force and is parallel to the last side.

Notice that the converse of the polygon of forces does not hold good, since two polygons may have their sides parallel and yet not proportional.

Ex. 1. Three forces acting at a point are represented by adjacent sides of a regular hexagon taken in order; find their resultant.

Ans. That diameter of the circum-circle which is parallel to the middle side.

Ex. 2. Find the resultant of four forces of 4, 5, 7 and 8 lbs., acting on a particle, and represented in direction by the successive sides of a square.

Ans. $3\sqrt{2}$ lbs. bisecting the angle between the forces of 7 lbs. and 8 lbs.

Ex. 3. In the hexagon $ABCDEF$, the lines AD, DE, EB, BC, CF represent in magnitude and direction five forces acting at a point; find their resultant. *Ans.* AF.

Ex. 4. Taking an inch to represent in magnitude a force of 1 lb. weight, by means of an ordinary foot-rule and a diagram, find the resultant of the following forces acting at a point:

$3\frac{1}{2}$ lbs. due E., 4 lbs. due S.E., 1 lb. due N.E., $6\frac{1}{3}$ lbs. due N.

52. Resolution of a Vector into Components.

A velocity may be resolved into two component velocities in an infinite number of ways, for the straight line representing the velocity may be the diagonal of an infinite number of parallelograms. Similarly a force can be resolved into equivalent pairs of component forces acting at the point.

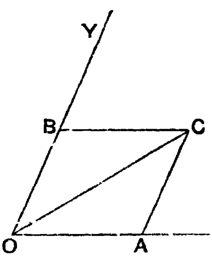

FIG. 28.

If the *directions* of the components are given, their magnitudes can be found at once.

Let OX, OY be the given directions, OC the given vector. Through C draw CA, CB parallel to OY, OX respectively. The OA and OB are the required components.

53. Resolved Part of a Vector.

In the important case when the directions OX, OY are at right angles, OA and OB are rectangular components of OC; they are called the *resolved parts* of OC in the directions OX, OY respectively. We see from the figure that the resolved part in any direction OX is represented by the projection of OC on that direction, that is, by $OC \cos COX$. Similarly the resolved part OB in the direction OY is equal to $OC \cos COY$, that is, $OC \sin COX$.

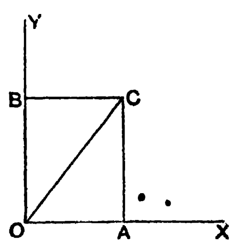

FIG. 29.

54. As an example of the resolution of a force into components take the case of the action of the wind on the sail of a vessel.

First resolve the force exerted by the wind into a component R perpendicular to the sail and another component along the sail, the latter component will have no effect. Next resolve R into components P and Q in the direction of the length of the vessel and perpendicular to it respectively.

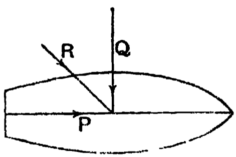

Fig. 30.

The component Q produces merely a slight broadside motion called *leeway*, the component P causes the vessel to move in the direction of its length. We easily see that by setting the sail so as to be nearly parallel to the direction of the wind the vessel may be made to go in a direction nearly opposite to that of the wind, neglecting the effect of leeway.

55. Resultant of any number of Velocities.

When a body has several simultaneous velocities in different directions its resultant velocity may be found :—

(i) By repeatedly using the parallelogram of velocities, viz., by finding the resultant of two velocities and then the resultant of this and a third velocity, and so on.

(ii) By the Polygon of Velocities.

(iii) A third method given in Art. 57.

It is sometimes useful to remember that since the diagonals of a parallelogram bisect each other, the resultant of two velocities OA and OB is $2 . OD$, where D is the middle point of AB.

56. The Resolved Resultant equals the sum of the Resolved Components.

Consider any number of component vectors acting in lines which pass through a point, for example, any component forces acting on a particle. From Art. 47, if lines are drawn in order representing the component vectors, the line which closes the polygon represents the resultant vector. Fig. 31 is a vector polygon for three components represented by PQ, QR, RS and the resultant is given by PS. Let AB be any given direction in which the vectors are to be resolved.

Now from Art. 53 the resolved part of any vector along AB is represented by the projection upon AB of the corresponding side of the polygon. From the figure we have

$$pq + qr + rs = ps.$$

Hence the sum of the resolved parts of the components is equal to the resolved part of the resultant in the same direction. Fig. 32 represents another case of three component vectors and their resultant, resolved along any given direction AB. Here the component vectors have the directions PQ, QR, RS; hence the resolved parts (which are also vector

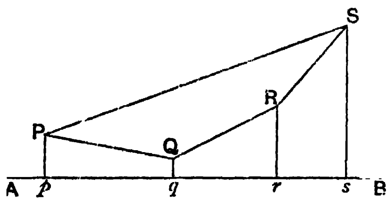

FIG. 31.

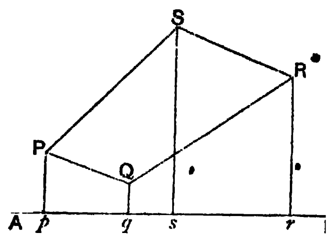

FIG. 32.

quantities) have the directions pq, qr, rs. In this case pq, qr are in one direction along AB, while rs is in the opposite direction; if the former are called positive, rs is negative. The *algebraic* sum of pq, qr and rs is equal to ps.

The resolved part of a vector along a line is also a vector; hence in forming the *sum* of the resolved components along any line, account must be taken of the direction of each along the line. The argument may be extended to any number of component vectors acting through a point, and we obtain the general theorem stated above.

57. Third method of finding the Resultant Velocity.

If α is the inclination of a velocity V to OX, then we have by the preceding;

resolved part of V along OX = projection of V on $OX = V\cos\alpha$,

.... OY = ... $OY = V\sin\alpha$.

If there are several velocities V, V', ... whose inclinations to OX are α, α', ..., the components of the resultant R are

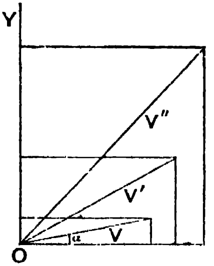

FIG. 33.

$$V\cos\alpha + V'\cos\alpha' + \ldots\ldots = X,$$
$$V\sin\alpha + V'\sin\alpha' + \ldots\ldots = Y.$$

Also $$R = \sqrt{X^2 + Y^2}.$$

If θ be the angle which R makes with OX,

$$\tan\theta = \frac{Y}{X} = \frac{V\sin\alpha + V'\sin\alpha' + \ldots}{V\cos\alpha + V'\cos\alpha' + \ldots}.$$

Ex. 1. Find the horizontal and vertical components of a velocity of V feet per second when inclined at an angle (i) of 30°, (ii) of 45°, (iii) of 60° to the horizon.

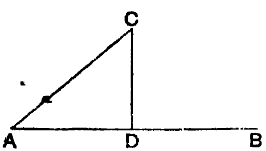

FIG. 34.

Let AC represent the given velocity in the three cases; draw CD perpendicular to AB, AD and DC represent the required components.

(i) When BAC is 30°. The angle ACD is 60°, and ADC is half an equilateral triangle, hence

$$AD = \frac{\sqrt{3}}{2}AC, \quad DC = \tfrac{1}{2}AC;$$

and the components are $$\frac{\sqrt{3}}{2}V, \quad \tfrac{1}{2}V.$$

(ii) When BAC is 45°. The angle ACD is also 45°, hence

$$AD = \frac{1}{\sqrt{2}}AC, \quad DC = \frac{1}{\sqrt{2}}AC;$$

the components are $$\frac{1}{\sqrt{2}}V, \quad \frac{1}{\sqrt{2}}V.$$

(iii) When BAC is 60°. The angle ACD is 30°, and

$$AD=\tfrac{1}{2}AC,\quad DC=\frac{\sqrt{3}}{2}AC;$$

the components are $\quad \tfrac{1}{2}V,\quad \frac{\sqrt{3}}{2}V.$

Ex. 2. Two velocities v_1 and v_2 in the directions OA and OB include an angle of 30°, find the resolved part of their resultant along a line which makes an angle of 60° with the direction of OA and 30° with the direction of OB.

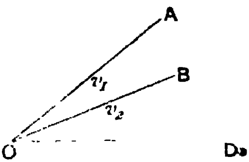

FIG. 35.

Drawing perpendiculars from A and B on the given line OD we see as before, since the angle between OA and OD is 60°,

the resolved part of v_1 along OD is $\tfrac{1}{2}v_1$;

and since the angle between OB and OD is 30°, the resolved part of v_2 along OD is $\frac{\sqrt{3}}{2}v_2$.

Hence resolved part of resultant = sum of resolved parts of OA and OB

$$=\tfrac{1}{2}(v_1+\sqrt{3}v_2).$$

Ex. 3. Velocities of 6, 7, and 8 feet per second are possessed by a body in directions making angles of 120° with each other; find their resultant.

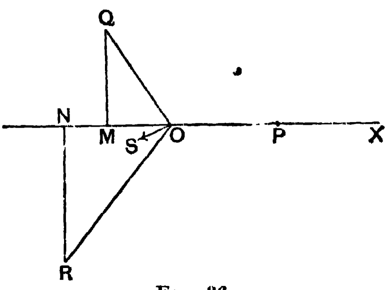

FIG. 36.

Take for the line OX of Art. 57 the direction of the velocity of 6 feet per second. Let OQ and OR represent the other two velocities. Draw QM and RN perpendicular to XO produced.

Then since

$$QOX=120°,\quad QOM=60°,\quad ROX=120°,\quad RON=60°,$$

the components along OX are

$$OP,\quad -\tfrac{1}{2}OQ,\quad -\tfrac{1}{2}OR.$$

The components perpendicular to OX are

$$\frac{\sqrt{3}}{2}OQ, \quad -\frac{\sqrt{3}}{2}OR.$$

The negative sign denotes as usual that a line is measured either to the *left* or *downwards*.

The component of the resultant along OX is therefore

$$OP-\tfrac{1}{2}OQ-\tfrac{1}{2}OR=6-\tfrac{7}{2}-\tfrac{8}{2}=-\tfrac{3}{2}.$$

The component of the resultant perpendicular to OX is

$$\frac{\sqrt{3}}{2}(7-8)=-\frac{\sqrt{3}}{2}.$$

If OS be their resultant,

$$OS^2=\tfrac{9}{4}+\tfrac{3}{4}=3 \text{ or } OS=\sqrt{3}.$$

The resultant velocity is one of $\sqrt{3}$ feet per second in the direction indicated in the figure.

Ex. 4. Four equal velocities each of magnitude u acting at O make angles α, β, γ, δ respectively with a line OA, such that

$$\cos\alpha=\tfrac{1}{3}, \quad \cos\beta=\tfrac{2}{7}, \quad \cos\gamma=-\tfrac{1}{4}, \quad \cos\delta=-\tfrac{1}{2}.$$

Find the resolved part of their resultant along OA.

The required sum is

$$u\times\tfrac{1}{3}+u\times\tfrac{2}{7}-u\times\tfrac{1}{4}-u\times\tfrac{1}{2}=\frac{28+24-21-42}{84}u=-\tfrac{11}{84}u.$$

Or a velocity of magnitude $\tfrac{11}{84}u$ in the direction AO.

EXAMPLES. XII.

1. Find the components of a velocity of 30 feet per second resolved along two lines inclined at angles of 30° and 60° respectively to its direction on opposite sides of it.

2. A body has a velocity of 40 feet per second in a direction which makes an angle of 45° with the horizon; find the horizontal and vertical components.

3. A boat is rowed across a river, flowing with a velocity of 3 miles an hour, so that the direction in which it is rowed makes an angle of 60° with either bank. If the velocity with which it is propelled be 8·8 feet per second, show that it will reach the other bank at the point immediately opposite that from which it started.

4. A boat is rowed across a river which flows at the rate of 2 miles per hour. If its breadth be 300 feet, find how far down the river the boat will reach the opposite bank below the point at which it was originally directed; the boat being propelled at the rate of 6 miles an hour.

5. Three velocities 12, 15, 24 are inclined at angles of 30°, 45°, 120° respectively to a given straight line; find the sum of their resolved parts along and perpendicular to it.

6. A point is moving with a velocity of 3 feet per minute along the diagonal of a square which is itself moving with a velocity of 4 feet per minute parallel to a side. Find the actual velocity of the point.

7. A ship is sailing due south at the rate of 4 feet per second; a current is carrying it due east at the rate of 3 feet per second, and a sailor is climbing up a vertical mast at the rate of 2 feet per second. What are the velocities of the ship and the man?

8. Four velocities of 30, 40, 50, and 60 feet per second make angles of 30°, 90°, 120°, 150° with a given line. Find their resultant.

9. Two velocities u and v make angles α, β, with a given straight line; show that their resultant is

$$\sqrt{u^2+v^2+2uv\cos(\alpha-\beta)}.$$

10. Three velocities p, q, r make angles α, β, γ with a given straight line; show that their resultant is

$$[p^2+q^2+r^2+2pq\cos(\alpha-\beta)+2qr\cos(\beta-\gamma)+2pr\cos(\alpha-\gamma)]^{\frac{1}{2}}.$$

58. Resultant of any Number of Forces.

As in the case of velocities we may find the resultant of any number of forces acting at a point in *either* of the following ways; viz., by use of

(i) repeated applications of the parallelogram-law;

(ii) the polygon of forces,

(iii) resolving the forces along two lines at right angles and then compounding the forces acting in these lines, see Art. 57.

The following examples show the application of the third method.

Ex. 1. The directions of three forces, acting at O, of 2, 3, and 5 lbs. make angles of 30°, 45°, and 60° respectively with a line OA, find their resultant.

Let OP, OQ and OR represent the forces, we resolve along and perpendicular to OA.

The components of OP along OA and OB are $2\times\frac{\sqrt{3}}{2}$, $2\times\frac{1}{2}$, p. 59.

.... OQ .... $3\times\frac{1}{\sqrt{2}}$, $3+\frac{1}{\sqrt{2}}$,

.... OR .... $5\times\frac{1}{2}$, $5\times\frac{\sqrt{3}}{2}$.

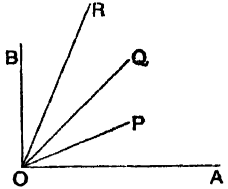

FIG. 37.

Hence the resolved parts of the forces along OA and OB respectively are

$$\sqrt{3}+\frac{3}{\sqrt{2}}+\frac{5}{2},\ 1+\frac{3}{\sqrt{2}}+\frac{5\sqrt{3}}{2}.$$

Therefore the resultant is

$$\sqrt{\left(\sqrt{3}+\frac{3}{\sqrt{2}}+\frac{5}{2}\right)^2+\left(1+\frac{3}{\sqrt{2}}+\frac{5\sqrt{3}}{2}\right)^2}$$

$$=\frac{1}{2}\sqrt{152+40\sqrt{3}+42(\sqrt{2}+\sqrt{6})}$$

$$=9{\cdot}8 \text{ lbs. nearly.}$$

Ex. 2. A particle is acted on by three forces of P, $2\sqrt{2}P$, $3\sqrt{2}P$ lbs., the angles between the first and second, and the second and third being 45° and 90° respectively; find the resultant.

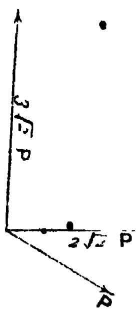

FIG. 38.

It shortens the work if we resolve the forces along and perpendicular to the directions of one of the forces themselves, let this be the force P.

Let X be the sum of components of all the forces in the direction of P, and Y be the sum of components of all the forces perpendicular to P.

Then $$X=P+2\sqrt{2}P\times\frac{1}{\sqrt{2}}-3\sqrt{2}P\times\frac{1}{\sqrt{2}}=0,$$

$$Y=2\sqrt{2}P\times\frac{1}{\sqrt{2}}+3\sqrt{2}P\times\frac{1}{\sqrt{2}}=5P.$$

Thus the resultant is perpendicular to the force P and equal to the weight of $5P$ lbs.

EXAMPLES. XIII.

1. Three equal forces each of 2 lbs. weight act on a particle. The angle between the directions of the first and second is 30°, between the second and third is 60°. Find their resultant. [Resolve the second force.]

2. A particle is acted on by three forces, one of 2 lbs. east, one of 2 lbs. north, one of $2\sqrt{2}$ lbs. north-west. Find the resultant.

3. Three forces act at a point, the angle between the first and second is 90°, and between the second and third is 60°. The second and third forces are each equal to F and the first is $\sqrt{3}F$. Find the resultant.

4. Forces of 1, 2 and $\sqrt{3}$ poundals act at a point O in the directions OA, OB and OC; the angle AOB is 60°, and the angle AOC is 90°, find the resultant.

5. Find the resultant of the following forces acting on a particle:

$3\frac{1}{2}$ lbs. due E., 4 lbs. due S.E., 1 lb. due N.E., $6\frac{1}{2}$ lbs. due N.

6. Four forces of 12, 10, 6 and 8 lbs. weight act on a particle. The angle between the first and second is 30°, between the second and third 120°, between the third and fourth 90°. Shew that the components along and perpendicular to the direction of the first force are

$$8+2\sqrt{3}\text{ lbs.},\ 8-4\sqrt{3}\text{ lbs.}$$

7. In the quadrilateral $ABCD$ the forces represented by AB, BD and DC have the same resultant as those represented by AD, DB and BC, the forces being supposed to act at a point.

8. Forces are represented by the radii of a circle drawn to the angular points of a regular inscribed polygon, show by the Polygon of Forces that these forces are in equilibrium.

9. ABC, $A'B'C'$ are two triangles in one plane, show that a hexagon can be constructed each of whose sides is equal and parallel to one of the sides of the two triangles.

10. Forces acting at a point are represented by the sides AB, BC, CD and the diagonal DB of a square, find the resultant force.

11. $ABCD$ is a parallelogram, show that the resultant of two forces represented by AC and DB is represented by $2AB$.

12. Forces of 2, 5 and 8 lbs. act parallel to consecutive sides of a regular hexagon taken in order; show that the sum of their components parallel to the middle force is one of 10 lbs.

13. $ABCD$ is a square and forces acting at a point are represented in magnitude and direction by AB, $2BC$, $2CD$ and $3DA$, what line represents their resultant?

14. Six forces act at a point parallel to the sides of a regular hexagon taken in order, the forces being of 3, 4, 6, 8, 10, and 11 lbs.; find their components parallel and perpendicular to the first force and show that their resultant is 11 lbs.

15. Forces P, Q, R, S acting at a point O are represented in direction by the sides AB, BC, CD, DA of a square taken in order. Find the magnitude of their resultant.

16. Forces P, Q, R act at a point and are parallel to the sides of an equilateral triangle taken in order. Find the magnitude of their resultant.

17. The angles between the directions of three forces in equilibrium are 120°, 150°, 90°, find the ratios of the forces.

18. Three forces respectively equal to 10 lbs. wt., 10 lbs. wt. and $10\sqrt{2}$ lbs. wt. are in equilibrium, find the angles between their directions.

19. Forces of 3 and $3\sqrt{3}$ lbs. wt. act at a point, the angle between them being 150°, find their resultant force and its inclinations to them.

20. Two forces of 3 lbs. and 4 lbs. wt. respectively, act at an angle of 60°. Find the sines of the angles their resultant makes with them.

21. A heavy particle is held at rest by means of two strings attached to it, one of which is horizontal. If the tension of one string is double that of the other, find the inclination to the vertical of the string which is not horizontal.

59. Relative Velocity.

Let two bodies A and B be moving in different directions, and with different velocities u and v.

The velocity of either viewed from the other will appear to be different both in magnitude and direction from either u or v.

To determine the motion of B as seen from A.

Apply to each body a velocity *equal and opposite* to that of A.

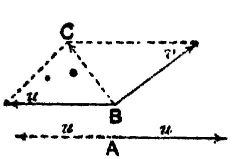

FIG. 39.

The *relative* motion will be unchanged, but A will be brought to rest, and B will have a resultant velocity along BC, *i.e.* B as seen from A will appear to move with a velocity represented in magnitude and direction by BC.

Thus in all cases the relative velocity of a point B, with reference to a point A, is found by combining with the velocity of B a velocity equal and opposite to that of A.

Ex. 1. A ship is steaming due south with a velocity of 10 knots, while another is steaming north-east at the rate of 15 knots. Find the velocity of the second ship with reference to the first.

Let a velocity equal and opposite to that of the first be given to each vessel; the first is brought to rest and the second has two velocities; viz.

a velocity of 10 knots northwards,
.... 15 knots north-eastwards.

The included angle is 45°, hence the resultant is

$$\sqrt{100+225+150\sqrt{2}} \text{ knots, or 23 knots nearly.}$$

Ex. 2. A ship is sailing due north at the rate of 7 miles an hour; in what apparent direction, as seen from the ship, and with what velocity must a man run on its deck that his actual direction may be due west and his actual velocity 7 miles an hour?

If the ship be brought to rest, the man will be moving due west with a velocity of 7 miles an hour, he has also a velocity of 7 miles an hour south. His apparent direction is therefore south-west and his relative velocity $7\sqrt{2}$ miles an hour.

EXAMPLES. XIV.

1. A railway train moving at the rate of 30 miles an hour passes another moving at the rate of 5 miles an hour in the same direction. Find the apparent velocity of the first train from the second train.

2. If the trains in the last question are going in opposite directions find the apparent velocity of either viewed from the other.

3. A train moving at the rate of 60 miles an hour is struck by a stone moving at right angles to the train with a velocity of 33 feet per second. Find the magnitude of the velocity with which the stone appears to strike the train.

4. Two straight lines of railway contain an angle of 60°; two engines run, one on each line, each from the point of intersection of the lines at the rate of 30 miles an hour. Find the magnitude of their relative velocities.

5. A ship is sailing north-east with a velocity of 10 miles an hour, and to a passenger on board the wind appears to blow from the north with a velocity of $10\sqrt{2}$ miles an hour. Find the true velocity of the wind.

6. The velocity of a ship in a straight course is $8\frac{1}{10}$ miles per hour, a ball is rolled across the deck perpendicular to the ship's length with a velocity of 3 yards in a second, show that it will pass over 5 yards in one second nearly.

7. A steamer is going due north with a velocity v, the smoke from its funnel points $\theta°$ south of east. If the wind is due west find its velocity.

8. A cricket-ball is moving in the line of wickets with a velocity of 30 feet per second and is struck by a blow which had the ball been at rest would have sent it with a velocity of 40 miles an hour at right angles to the line of wickets. In what direction will it go?

9. A company of soldiers is marching along a road at the rate of 3 miles an hour, the column is 3 yards wide and there is just room for one man between two consecutive ranks. A man crosses over from one side of the column to the other walking at the rate of 5 miles an hour. In what direction does he walk and how long does he take to cross over?

10. Two men A, B walk along two straight paths at right angles to one another with velocities of 8 feet per second and 6 feet per second respectively. A passes the intersection of the paths at the instant when B has still 100 feet to go to reach it. After what interval of time will the distance between A and B be again 100 feet?

11. Two motor cars are moving uniformly on two straight roads perpendicular to each other at 40 and 20 miles per hour respectively. At a certain instant they are both 5 miles from the point of intersection of their paths and are moving towards it; how much time will elapse before they are at their shortest distance from each other, and what is that shortest distance?

60. Motion down an Inclined Plane.

As an example of the resolution of forces take the case of a body which falls down a smooth inclined plane.

The forces acting on it are its weight W which is vertical and the reaction of the plane R which is perpendicular to the plane.

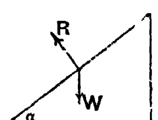

Fig. 40.

W may be resolved into two components at right angles to each other,

$W\cos\alpha$ perpendicular to the plane,

$W\sin\alpha$ along the plane.

The force $W\cos\alpha$ is balanced by R, hence

$$W\cos\alpha = R,$$

because there is no motion perpendicular to the plane; there remains a force $W \sin \alpha$ or $mg \sin \alpha$ along the plane.

The *acceleration* along the plane is therefore $\frac{mg \sin \alpha}{m}$ or $g \sin \alpha$, Art. 35.

If v is the velocity gained by falling down the plane a distance s,

$$v^2 = 2g \sin \alpha \,.\, s \text{..................Art. 19}$$

$$= 2gh,$$

where $h = s \,.\, \sin \alpha =$ vertical height fallen through.

Hence the velocity gained by falling a *vertical height* h, is always the same, whatever the inclination α of the plane may be.

Ex. Two bodies, one on each face of a double inclined plane, are connected by an inextensible string, which passes over the vertex; to find the motion.

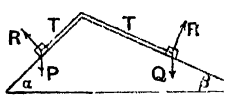

FIG. 41.

Let T be the tension of the string and P and Q the weights of the bodies, f the acceleration of *each* body along the plane.

Suppose P to be the body which descends and Q that which ascends; the forces on the bodies along the planes in the direction of motion are

$$P \sin \alpha - T \text{ and } T - Q \sin \beta.$$

Therefore
$$\frac{P}{g} f = P \sin \alpha - T,$$

$$\frac{Q}{g} f = T - Q \sin \beta;$$

hence
$$\frac{P+Q}{g} f = P \sin \alpha - Q \sin \beta,$$

or
$$f = \frac{P \sin \alpha - Q \sin \beta}{P + Q} g.$$

61. Motion down Chords of a Vertical Circle.

Let a body whose mass is m slide down a chord AP of a circle in a vertical plane, starting from A the highest point. Let the angle which AP makes with the vertical be θ.

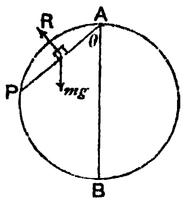

FIG. 42.

The force acting on the body in the direction of AP is $mg\cos\theta$.

The body's acceleration along AP is therefore $g\cos\theta$.

Hence if the body starts from rest at A and takes a time t to reach P

$$AP = \tfrac{1}{2} g \cos\theta \,.\, t^2.$$

But $$AP = AB\cos\theta,$$

$$\therefore \quad AB\cos\theta = \tfrac{1}{2} g\cos\theta \,.\, t^2,$$

or $$t^2 = \frac{2AB}{g}, \quad \text{and} \quad t = \sqrt{\frac{2AB}{g}}.$$

Thus the time taken to reach P does not depend upon θ and is therefore the same for *any other* chord drawn through A.

We thus see that the time taken by a body to slide down *any* chord of a vertical circle starting from the highest point is the *same*, and equal to that taken to fall freely through the distance AB, the diameter of the circle.

EXAMPLES. XV.

1. The sides of a quadrilateral are 1, 3, 5, 6. Forces of 2, 6, 9 and 12 lbs. wt. respectively act on a particle parallel to the sides of the quadrilateral taken in order. Find their resultant.

2. If the component of a force P in the direction OA is equal to that of Q in the same direction, and if their components in another direction OB are also equal, prove that P is equal to Q.

3. Three forces acting at one point are in equilibrium, one of them is turned round this point through a given angle, find the direction of the resultant of the three forces.

4. A train weighing 200 tons is running at 40 miles an hour down an incline of 1 in 20, find the resistance necessary to stop it in half a mile.

5. A train ascends a gradient of 1 in 40 by its own momentum for a distance of $\frac{1}{4}$ mile and then stops, the resistance from friction, etc. being 10 lbs. per ton and the weight of the train 250 tons, find its initial velocity.

6. A stone leaves the top of a tower 320 feet high with the velocity acquired by sliding down an inclined plane (of inclination 30°) for a distance of 32 feet. Show that it strikes the ground about 111 feet from the foot of the tower.

7. Find what force a horse has to exert to prevent a railway truck weighing 5 tons from descending a smooth incline of 1 in 300.

8. If the resultant of two forces PA, PB pass through a point Q, the resultant of the forces QA, QB will pass through P.

9. If the greatest possible resultant of two forces P and Q is m times the least possible, their inclination when their resultant is $\frac{1}{2}$ their sum is a, where $\cos a = -\dfrac{m^2+2}{2(m^2-1)}$.

10. A weight is supported by two strings fastened to two points in the same horizontal line, the strings being equally strong, but one rather longer than the other. If more weights be continually added to the first one which string will break first?

11. The resultant of two forces of 12 lbs. and 5 lbs. weight is 13 lbs. weight, what will the resultant be if the forces receive an increase in *magnitude* of 3 lbs. weight?

12. At what angle must the forces $A+B$ and $A-B$ act, in order that their resultant may be $\sqrt{A^2+3B^2}$?

13. A body P is attracted towards the points A and C the opposite extremities of the diagonal AC of the parallelogram $ABCD$ by forces proportional to PA and PC respectively, and is repelled from B and D by forces proportional to PB and PD. Show that it is in equilibrium wherever P be situated.

14. A heavy particle of weight W is supported on an inclined plane, whose inclination to the horizon is a, by 3 forces each equal to P tending upwards and acting respectively along the plane and making angles a and $2a$ with it. If $\sin a = \frac{3}{5}$ show that P is equal to $\frac{1}{5}\frac{5}{2}W$, and that the pressure on the plane is $\frac{7}{20}W$.

15. A system of forces acting on a particle is represented by straight lines in magnitude equal and in direction parallel to straight lines drawn from the angles of a quadrilateral to the middle points of each of the opposite sides. Prove that the forces are in equilibrium.

16. Resolve a force P acting along the diagonal of a given square into two components acting along the straight lines joining one end of that diagonal with the middle points of the opposite sides.

17. A body starting from rest down an inclined plane describes 40 feet in the third second, find the plane's inclination.

18. A weight of W lbs. is drawn from rest up a smooth inclined plane of height h and length l, by means of a string passing over a pulley at the top of the plane and supporting a weight of w lbs. hanging freely. Prove that in order that W may just reach the top of the plane, w must be detached after it has descended a distance

$$\frac{W+w}{w}\,\frac{hl}{h+l}.$$

19. A and B are two fixed points on a circle, P a point on the circumference, if two constant forces act along PA, PB, prove their resultant is constant in magnitude and passes through a fixed point.

20. At the same moment two particles begin to slide down two smooth straight lines in the same vertical plane which slope towards the same direction and are inclined at angles θ and ϕ to the horizon. Show that each particle as seen from the other will appear to move parallel to a line inclined to the horizon at an angle $\theta+\phi$ towards the same direction.

CHAPTER IV.

MOMENTUM, WORK AND ENERGY.

62. Momentum.

Suppose we observe a body of mass m which starts from rest and moves in a straight line for a time t under the action of a constant force F absolute units, attaining a velocity v at the end of the time; we know that the relation connecting these quantities is

$$Ft = mv \quad \text{..........................(1)}.$$

The product Ft is called the total impulse of F acting for a time t, hence the equation may be written

total impulse of a force = gain of momentum,

both products being measured in the same absolute units of momentum. Conversely, if a mass m is moving with velocity v, equation (1) gives the *time* for which it will move before being brought to rest under the action of a constant retarding force F.

62*a*. Conservation of Momentum.

Consider the illustration which was used in Art. 41. A shot of mass m is projected horizontally from a gun of mass M; the shot leaves the barrel with velocity v, while the velocity of recoil of the gun is V. The explosive force of the charge acts equally forwards on the shot and backwards on the gun for the same time, hence the products MV and mv are equal numerically.

We may regard the gun and shot as forming one system; then the force of the charge is an internal force giving rise

to equal action and reaction on two parts of the system, and during the time under consideration there is no external horizontal force acting on the system as a whole. Consequently there is no change of total momentum of the system horizontally; remembering that the velocities V and v are measured in opposite directions, we have

$$MV - mv = 0.$$

This example is a simple case of a general principle which may be stated as follows: *If the sum of the external forces acting on any system resolved in any direction is always zero, the total momentum of the system in that direction remains constant during the motion.*

Ex. 1. A hammer weighing 10 cwt. falls through 4 feet and comes to rest after striking a mass of iron; the duration of the blow is $\frac{1}{5}$ second. Find the pressure, supposing it uniform, which the hammer exerts upon the iron. *Ans.* 2800 lbs. wt.

Ex. 2. A gun of 15 tons mass fires a shot of 1 cwt. horizontally with a velocity relative to the muzzle of 1000 feet per second; find the velocity of the gun's recoil. *Ans.* 3·34 feet per second.

Ex. 3. Two bodies moving in opposite directions with velocities of 8 feet per second and 10 feet per second impinge directly, and then adhering move with a velocity of 2 feet per second in the direction of the first body; compare the masses of the two bodies. *Ans.* 1 : 2.

63. Energy.

We obtain now a similar relation involving the distance s instead of the time t. If the mass m starts from rest and covers a distance s, attaining a velocity v at the end of this distance, we know that v, s, and the acceleration a are connected by

$$2as = v^2.$$

Hence also $$mas = \tfrac{1}{2}mv^2 \qquad \text{.....................(2)}.$$

But if F is the force acting on the body, we have

$$F = ma \qquad \text{.....................(3)}.$$

Hence $$Fs = \tfrac{1}{2}mv^2 \qquad \text{.....................(4)},$$

both products being in the same absolute units. There are special names for both these products.

Fs is called the Work done by the force F when its point of application is moved a distance s in the direction of F.

$\frac{1}{2}mv^2$ is called the Kinetic Energy of a mass m which is moving with velocity v.

Hence in this simple case we have the equation

Work done = gain of kinetic energy.

Conversely, if we have a mass m moving with velocity v, equation (4) gives the *distance* for which it will move before being brought to rest by a constant retarding force of F absolute units.

64. Units of Work and Energy.

Unit work is done by unit force in moving the body on which it acts through unit distance in its own direction.

In scientific measurements the C.G.S. units of length, mass, and time are used. The unit of work is the Erg, which is the work done by a force of one Dyne moving its point of application through one centimetre; if the amount of work is large it may be expressed in terms of the Joule, which is 10^7 Ergs.

If in equation (4) we use English measure of the pound, foot, and second, then F must be in poundals, and the product Fs is said to be in foot-poundals; but this unit is not used in practice.

To obtain the practical unit of work in English units we measure the force F in lbs. weight; then instead of substituting ma for F in (2) we have to use the relation

$$F = \frac{ma}{32{\cdot}2}.$$

Hence we have

$$Fs = \frac{1}{2}\frac{m}{32{\cdot}2}v^2 \text{ ft.-lbs.} \quad \ldots\ldots\ldots\ldots\ldots(5).$$

The English unit of work is the foot-pound, the work done by a force of one lb.-weight moving its point of application through one foot in its own direction.

We have then

$$\begin{aligned} 1 \text{ ft.-lb.} &= g \text{ foot-poundals} \\ &= 32 \text{ ft.-poundals, approximately.} \end{aligned}$$

EXAMPLES. XVI.

1. A man weighing 12 stone goes up a staircase of 30 steps so as to rise one foot vertically each step. How many foot-pounds of work must he do before reaching the top?

2. Show that the work done in raising 1 cwt. through a height of 10 yards is equal to the work done in raising 1 lb. a height 3360 feet.

3. Find the work done in drawing up a Venetian blind, the number of bars being 50, the distance between each bar three inches, and the weight of each bar four ounces.

4. Find the work done by a force which acts for two seconds on a body whose mass is m lbs. and gives it a velocity of 10 feet per second. [$Fs = ma \times \frac{1}{2}at^2$.]

5. A gun whose weight is one ton is drawn 100 feet along the ground; if the resistance due to the roughness of the ground is $\frac{1}{10}$th of the weight of the gun, find the work done against the resistance.

65. Work of Component Forces.

Given a mass m under the action of two component forces meeting at a point it will move as if under the action of a single force R making angles α, β with P, Q to be determined by means of the parallelogram law. Let the body start from O and move a distance s along the line of R; in the figure we have OC equal to s and CM, CN perpendicular to the lines OP, OQ.

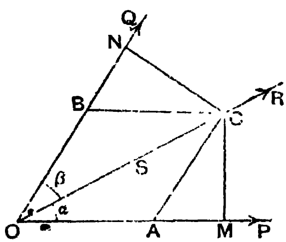

FIG. 43.

Then we have

$$\begin{aligned}\text{Work done in displacement} &= \text{work of resultant force } R \\ &= Rs \\ &= (P\cos\alpha + Q\cos\beta) \,.\, s \\ &= P \,.\, s\cos\alpha + Q \,.\, s\cos\beta \\ &= P \,.\, OM + Q \,.\, ON.\end{aligned}$$

If we define the products $P \times OM$ and $Q \times ON$ as the work done by the component forces in the actual displacement, we shall have

Work of resultant force R = sum of works of components P, Q

Definition. The work of a force P acting on a body when its point of application is moved a distance S in any direction is the product of the force into the resolved part of the displacement in the direction of the force.

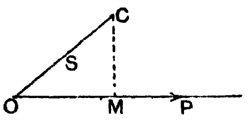

FIG. 44.

If the displacement is in the same line as the force but in the opposite direction, the work done is said to be negative; or, work is done against the force.

If the point of application is moved at right angles to the force, then no work is done either by or against that component force.

66. Work done in falling down Inclined Plane.

When a body falls down a smooth inclined plane, the resistance of the plane being perpendicular to the plane does no work. Art. 65.

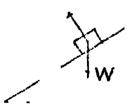

FIG. 45.

The work done by gravity is equal to $W.h$, where W is the weight of the body and h the height of the plane, since h is the projection of the displacement on th direction of gravity.

Hence the work done in descending an inclined plane depends only on the weight of the body and the height of the plane.

67. Another Expression for Work done.

From Art. 65 we see that the work done by a force F is $F \times AC$, when its point of application moves a distance AC in the direction of F.

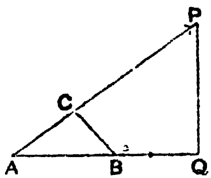

FIG. 46.

Let AP represent the force and AB the displacement, draw BC perpendicular to AP and PQ perpendicular to AB produced.

Then B, C, P, Q are four points on the same circle.

Hence $AP \times AC = AB \times AQ$. Euc. III. 36.

But $AP \times AC$, or $F \times AC$, is the work done, also AQ is the component of F in the direction of the displacement,

hence work done is = displacement × component of F in the direction of the displacement.

68. Work of Resultant equals sum of Works of Components.

By the previous Article, when any number of forces act at a point which undergoes a displacement, the total work done

= displacement × (sum of components of forces in direction of displacement)

= displacement × component of resultant of the forces in direction of displacement

= work of resultant of the forces.

69. Principle of Virtual Work.

If, in the last Article, the resultant is zero, or the forces are in equilibrium, we see that the total work done by the forces is zero.

When the displacement is not actual but it is merely *supposed* that the point of application of the forces receives a displacement, the work done by any of the forces is said to be *virtual*, and the displacement is called a *virtual* displacement.

The Principle of Virtual Work states that when a set of forces is in equilibrium the algebraic sum of the virtual work of all the forces is zero, the forces being supposed to remain the same during the displacement. To secure that the forces may not sensibly alter during the virtual displacement it is usually taken *very small*.

The displacements are taken so that the work done by the forces which we do not wish to find does not appear, for instance if we take the displacement of a rigid body such that the distances between its particles are not altered, the internal forces (see Art. 78) will do no work. If the displacement consists of a small rotation round any point in the plane of the body, it is easy to see that, if the forces are in equilibrium, we thus get the equation of *moments* round the given point.

70. Rate of Work.

The total actual work Fs performed by F in a displacement s in its own direction does not depend upon the time

taken to do it, but if we suppose it to be performed uniformly in a certain time t, we obtain the rate of doing work, namely

$$\frac{Fs}{t}, \text{ or } Fv \text{ units of work per unit time.}$$

The rate of work is sometimes called the Power or the Activity of the agent or machine which supplies the operating forces.

In the C.G.S. system the unit of Power is the Watt, or one Joule per second; that is

$$1 \text{ Watt} = 10^7 \text{ Ergs per second.}$$

In English units, we have the unit of a foot-lb. per second. One Horse Power is defined as 33,000 foot-lbs. per minute, or

$$1 \text{ H.P.} = 550 \text{ foot-lbs. per second.}$$

Ex. 1. A train of 100 tons is pulled by a locomotive on the level at a constant speed of 30 miles per hour, the resistance amounts to 15 lbs. per ton. Find the minimum horse-power of the engine.

The resistance to motion is 1500 lbs.; and a speed of 30 miles per hour is 44 feet per second.

Therefore the rate at which the engine works is 1500×44 foot-pounds per second.

Hence minimum horse-power is $\frac{1500 \times 44}{550}$, or 120.

Ex. 2. If the train described in the last example be moving at a particular instant with a velocity of 15 miles an hour, what is the acceleration at that instant?

The engine is doing 120×550 foot-pounds of work per second, while the train is moving at the rate of 22 feet per second.

Therefore the force of the engine, or $\frac{550H}{v}$, is here equal to $\frac{120 \times 550}{22}$ pounds' weight, or 3000 lbs. weight.

Of this 1500 lbs. weight is needed to overcome the resistance, hence the remainder 1500 lbs. weight is the effective force,

therefore effective force is 1500×32 poundals,

$$\text{acceleration} = \frac{1500 \times 32}{\text{mass of train}} = \frac{1500 \times 32}{2240 \times 100} = \frac{3}{14}.$$

EXAMPLES. XVII.

1. The mass of a complete train is 60 tons, and the resistance to its motion equal to 20 lbs. weight per ton. The locomotive has 240 horse-power; what is the highest speed the train can have?

2. An engine is required to raise in 4 minutes a weight of 12 cwt. from a pit whose depth is 600 feet. Find the horse-power of the engine.

3. An engine draws a train weighing 96 tons at the rate of 15 miles an hour, the resistance to the motion of the train amounting to 8 lbs. per ton, find the horse-power of the engine.

4. Determine the rate in H.P. at which an engine must be able to work to generate a velocity of 30 miles an hour on the level in a train of mass 60 tons in 3 minutes after starting, the resistance to motion being 10 lbs. per ton weight.

71. Kinetic and Potential Energy.

We have seen that a body of mass m moving with velocity v can do a certain amount of work, namely $\frac{1}{2}mv^2$ absolute units, against a retarding force before it is brought to rest; thus kinetic energy represents capacity for doing work due to the motion of the body.

But there are other ways in which a body may possess energy, or capacity for doing work. If we raise a weight any height we can obtain work from it by allowing it to fall under gravity; again, if a spring is compressed we can obtain work by allowing it to unwind itself. In both cases there is a capacity for doing work due to the position or configurations of the bodies in question. Suppose in every case we choose some standard position or arrangement of the parts of the system; then the amount of work which is done by allowing the system to move from any given position to the standard position is called its Potential Energy in the given position.

For example, the potential energy of a weight of w lbs. which has been raised h feet from the Earth's surface is wh foot-lbs., if we choose the ground as the standard position; for if the body is allowed to fall to the ground, its weight performs wh foot-lbs. of work.

Let u be the velocity of the body after it has fallen any distance x feet.

Then since $$u^2 = 2gx,$$
we have, identically, from this equation

$$\frac{1}{2}\frac{w}{g}u^2 + w(h-x) = wh.$$

But the first term is the kinetic energy after falling x feet, and the second term is the potential energy at the same point; also the right hand side is the potential energy at the starting point. Hence we have at any instant during the fall of the body

Sum of kinetic energy and potential energy
= potential energy at the starting point
= constant quantity during the motion.

As the body falls its potential energy diminishes and is transformed into kinetic energy, so that the total energy is constant.

72. Conservation of Energy.

This example is the simplest case of an important general principle which may be stated thus:—If a body, or system of bodies, be under the action of forces which depend only on the position of the body, or system of bodies, the sum of the potential and kinetic energies is constant.

Ex. 1. A body whose weight is 5 lbs. is thrown vertically upwards with a velocity of 32 feet per second; find its K. E.,

(i) at the moment of projection, (ii) after half a second, (iii) after one second.

(i) At the moment of projection the velocity is 32, hence

$$\text{K.E.} = \frac{1}{2}\frac{w}{g}v^2 = 80 \text{ ft.-lbs.}$$

(ii) After half a second the velocity is 16, ∴ K.E. = 20 ft.-lbs.

(iii) After one second the velocity is 0, ∴ K.E. = 0.

Ex. 2. A ball weighing 6 lbs. is rolled on a floor with a velocity of 8 feet per second; if the resistance of the floor to the motion of the ball is $\frac{1}{10}$th of the pressure on it, how far will the ball roll?

If s be the required distance, the work done against the resistance is $\frac{6}{10} \times g \times s$, and this is equal to the K. E. of the ball, hence

$$\tfrac{6}{10} \times 32 \times s = \tfrac{1}{2} \times 6 \times 8^2;$$
$$\therefore \quad s = 10 \text{ feet}.$$

Ex. 3. A cannon-ball whose mass is 60 lbs. falls through a vertical distance of 400 feet; what is its kinetic energy?

The velocity acquired in falling through 400 feet is 160 feet per second

$$\therefore \quad \text{K. E.} = \tfrac{1}{2} \times 60 \times (160)^2$$
$$= 768000 \text{ ft.-poundals}$$
$$= 23851 \text{ ft.-lbs.}$$

EXAMPLES. XVIII.

1. A fourteen-ton gun on being fired recoils and is brought to rest by a uniform resistance equal to the weight of 3 tons. How far does the gun recoil, the velocity of the ball being 1200 feet per second and its mass 112 lbs.?

2. A ball whose mass is 100 grammes is thrown vertically upwards with a velocity of 980 centimetres per second; what is the kinetic energy of the body

   (i) at the moment of propulsion,
   (ii) after half a second,
   (iii) after one second?

3. A shot of 1000 lbs. moving 1600 feet per second strikes a fixed target; how far will the shot penetrate the target which exerts upon it an average pressure equal to the weight of 12,000 tons?

4. A ball whose mass is 10,000 grammes is discharged with a velocity of 6000 centimetres per second; find its K. E. in ergs.

5. A ball whose mass is 3 lbs. is moving at the rate of 100 feet per second; what force will stop it (i) in 2 seconds, (ii) in 2 feet?

6. The mass of a fly-wheel is 1200 kilogrammes and the radius of its rim one metre. Supposing the whole mass concentrated in the rim, find the energy of the wheel when making 7 turns a second.

73. Work represented by an Area.

In Article 21 we saw how to draw a velocity-time curve which gave the space covered as an area; we proceed to consider in an exactly similar way a force-distance diagram which will give the work done as an area.

74. Constant Force.

Given a constant force F lbs. wt. acting on a body and moving it through s feet, we know that the work done is Fs foot-lbs. Along a line OX take a length ON representing to scale the distance covered at any time from the start, then draw NP perpendicular to OX representing the force F in magnitude; since F is constant all the points P so found will lie on a straight line BC parallel to OX. This line is the force-distance curve in this case. Since $OA = s$, and $AC = F$, then the area $OACB = Fs$; thus the work done by F in the operation is represented by the area contained within the force-distance curve, the base line OX, and the initial and final ordinates OB, AC.

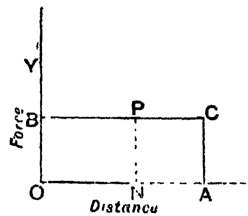

FIG. 47.

75. Uniformly increasing Force.

Given a body acted on by a force which increases at a uniform rate with the *distance* moved by the body from the initial position; that is if the force is F after the body has been moved a certain distance it is $2F$ after it has moved twice the distance, and so on. (Notice that in estimating work done we are not concerned with the variation of the distance or force with the time, but with the dependence of the force upon the distance.) Suppose that at the beginning the force is zero, and that finally after the body has moved a distance s the force is F'.

Then we take OA representing s to scale, and draw AC at right angles to represent F'. At any intermediate stage let ON be the distance travelled and NP the corresponding value of the force.

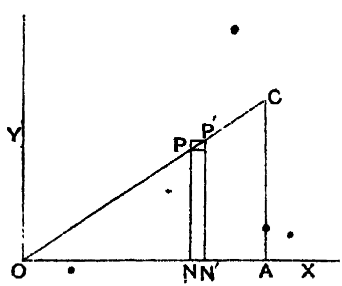

FIG. 48.

Then we know that PN increases uniformly with ON; that is the ratio PN/ON is constant at all stages. Hence the end-points P all lie on the straight line OC, which is consequently the force-distance curve in this case.

Consider the work done while any small distance NN' is covered; from Article 74 it is represented by less than the rectangle $P'N$ and by more than the rectangle PN'. Now divide the whole distance s into small parts and draw the similar pairs of rectangles. Then the whole work done is less than the sum of the outer rectangles and greater than the sum of the inner rectangles.

Supposing OA to be divided into increasingly smaller parts, we see, as in Art. 16, that ultimately the areas of the two sets of rectangles both approximate to the same area, namely the triangle OAC.

Hence

$$\text{Work done in the displacement} = \text{area } OAC$$
$$= \tfrac{1}{2} OA \times AC = \tfrac{1}{2} F's$$
$$= \tfrac{1}{2}\,(\text{final value of force}) \times (\text{total displacement}) \ldots (1).$$

An illustration of this is to be found in an elastic string, or a spiral spring. It has been found that if such a string is stretched a distance x beyond its natural length a, the tension or pull in the string is directly proportional to x (up to the elastic limit of the string); this is known as *Hooke's Law*. Let T be the tension at any stage, then it is usual to write

$$T = \lambda \frac{x}{a} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where λ is called the modulus of elasticity; λ depends upon the material of the string and would be the amount of the tension if the string were stretched to twice its natural length, provided the law held as far.

Then by (1) the work done in stretching a string (or similar spring) from its natural length a to a length $a + x$ is equal to

$$\tfrac{1}{2} Tx = \frac{1}{2} \frac{\lambda}{a} x^2.$$

This amount of work could be obtained by allowing the string to contract again, consequently this expression gives the potential energy of a stretched elastic string.

76. Variable Force in general.

Suppose a body is moving in a straight line under the action of a variable force and we can measure the force at a large number of positions during the displacement; then plotting the force against the displacement for these positions we obtain a number of end-points which can be joined by a curve, giving a force-distance curve such as DPC. It can be shown that in general the work done during the displacement AB is given by the area between the curve DPC, the base line AB, and the initial and final ordinates AD, BC.

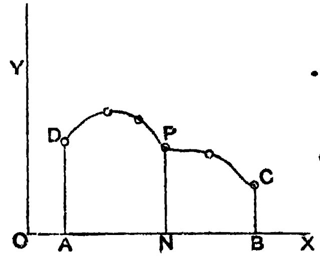

FIG. 49.

An example is an indicator diagram for a steam engine. Let AB represent the stroke of the piston. At any position N on the out-stroke let PN represent the total steam pressure

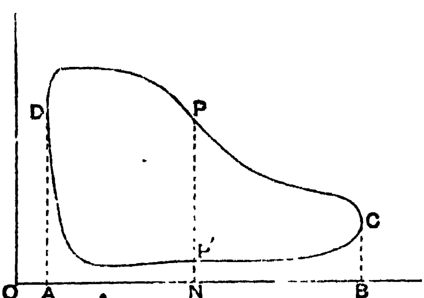

FIG. 50.

on the piston. Then the area $DPCBAD$ represents the work done by the expansion of the steam. Let $P'N$ represent the pressure at the same position on the return stroke; then

$CP'DABC$ is the work which has to be done against the steam pressure in the return stroke.

Hence the work gained in one complete revolution is represented by the difference of these two areas, that is, by the area $DPCP'D$. If we can measure this area and if we know the time of a revolution, we have the average rate at which work is being produced by the steam pressure. This is called the Indicated Horse Power; a certain proportion of this work is transformed into useful work, depending upon the efficiency of the engine.

EXAMPLES. XIX.

1. A spring is found to extend a distance of 6·2 inches beyond its natural length under a load of 20 lbs. Find the work done in gradually stretching the spring from its natural length to an extension of 3 inches.

The tension in the spring is proportional to the extension, hence the final tension for a gradual extension up to 3 inches is $\frac{3}{6 \cdot 2} \times 20$ lbs. wt. The tension increases uniformly from zero up to this amount, therefore the work done is $\frac{1}{2} \times \left(\frac{3}{6 \cdot 2} \times 20\right) \times \frac{3}{12}$ ft.-lbs. $= 1 \cdot 2$ ft.-lbs. approximately.

Draw a work diagram, and obtain the additional work required to stretch the spring another inch.

2. If a weight of 13·4 lbs. stretches a spring 6 inches beyond its natural length, find the work done in gradually extending it from an extension of 3 inches to one of 7 inches.

3. A weight is lifted by drawing up a uniform rope attached to it; show how to draw a diagram representing the work done against gravity.

4. Estimate, by drawing a diagram, the work done in diminishing the distance between two electrified particles (in air) from 8 cm. to 4 cm., the charges being like and equal to 10, 20 units respectively. (The force of repulsion between two like charges m, m' at a distance r cm. $= \frac{mm'}{r^2}$ dynes, in air.)

77. Units and Dimensions.

To measure any physical quantity, we must first choose some standard quantity of the same kind which we use as a unit; then the numerical measure is the ratio of the given quantity to the selected unit quantity. Thus if l is a given length and we choose a length L as unit, then the measure

of the given length is the ratio l/L. Similarly if t is a given time and T the unit of time, the measure of t is the ratio t/T; and if m, M are similar quantities for mass, the measure of the mass is m/M. We see that the measure of a quantity varies inversely as the quantity which is selected as a unit. In the measurement of all physical quantities the standard units for these three quantities—mass, length, and time are called fundamental units. For if we consider any of the quantities, such as velocity, force, energy, for which we have defined an absolute unit, we notice that in each case the only quantities referred to have been unit mass, unit length and unit time. All these other units are called derived units and we have to find how a derived unit varies when the fundamental units of mass, length, and time are altered in any given manner.

Defining unit area as the square on unit length, we see that if L is the unit length, then L^2 is the unit area.

Thus the unit area varies directly as the square of the unit of length, and the measure of area will vary inversely as the square of L. This is expressed shortly by saying that the *Dimensions* of area are (length)^2, and this is written shortly as

$$Area = L^2.$$

Similarly the dimensions of volume are written as

$$Volume = L^3.$$

That is, the unit of volume varies as the cube of the unit of length, and consequently the measure of a volume varies inversely as the cube of the unit of length.

Unit velocity being defined as unit length described in unit time, it is clear that if the unit length is changed in any ratio the unit velocity is changed in the same ratio, while if the unit of time is varied the unit velocity varies in the inverse ratio; hence we have the dimensional equation

$$Velocity = \frac{L}{T}.$$

Unit acceleration is unit velocity gained in unit time, hence we have

$$Acceleration = \frac{\text{velocity}}{\text{time}} = \frac{L}{T^2},$$

or the unit of acceleration varies directly as the unit of length and inversely as the square of the unit of time.

Unit force occurs for unit mass moving with unit acceleration, hence the unit varies directly as the unit of mass and the unit of acceleration; that is,

$$Force = (\text{mass}) \times (\text{acceleration}) = \frac{ML}{T^2}.$$

Similarly, from the definitions we have given previously of the several unit quantities we can write down the dimensional equations.

$$Momentum = (\text{mass}) \times (\text{velocity}) = \frac{ML}{T},$$

$$Energy = \text{work} = (\text{force}) \times (\text{length}) = \frac{ML^2}{T^2},$$

$$Power = \frac{\text{work}}{\text{time}} = \frac{ML^2}{T^3}.$$

Instead of using a separate name for any derived unit with given units of mass, length and time it is often convenient to write the dimensions of the quantity; for instance

$$\text{Unit acceleration, in C.G.S. units, } = 1\,\frac{\text{cm.}}{(\text{sec.})^2},$$

$$\text{Unit force} = 1 \text{ dyne} = 1\,\frac{\text{gr. cm.}}{(\text{sec.})^2},$$

$$1 \text{ poundal} = 1\,\frac{\text{lb.-ft.}}{(\text{sec.})^2}.$$

Ex. 1. A certain acceleration has as its measure 11 when 4 feet and 6 seconds are taken as units, find its measure in foot-second units.

Let x be the measure in the new units; then since the quantity is the same whatever the units, we have

$$x\,\frac{\text{ft.}}{(\text{sec.})^2} = 11\,\frac{4 \text{ ft.}}{(6 \text{ sec.})^2}$$

$$= \frac{44}{36}\,\frac{\text{ft.}}{(\text{sec.})^2},$$

$$\therefore\ x = \frac{11}{9}.$$

Ex. 2. Compare the values of the two units of force, a poundal and a dyne, given

$$1 \text{ lb.} = 453{\cdot}59 \text{ gr.}; \quad 1 \text{ ft.} = 30{\cdot}48 \text{ cm.}$$

We have $$x\frac{\text{gr. cm.}}{(\text{sec.})^2} = 1\frac{\text{lb.-ft.}}{(\text{sec.})^2} = 1\frac{(453{\cdot}59 \text{ gr.})(30{\cdot}48 \text{ cm.})}{(\text{sec.})^2}$$

$$= 13825\frac{\text{gr. cm.}}{(\text{sec.})^2}, \text{ approximately.}$$

$$\therefore\ 13825 \text{ dynes} = 1 \text{ poundal.}$$

EXAMPLES. XX.

1. What is the measure of a velocity of one foot per second when a yard and an hour are the units of length and time respectively?

2. Find the measure of the acceleration of gravity when the units of length and time are a mile and a minute.

3. If the weight of a lb. is the unit of force, a velocity of one yard per second the unit of velocity, the mass of 4 lbs. the unit of mass, find the units of length and time.

4. The units of length and time being the same as in question 1, find the measure of a poundal, the unit mass being one lb.

5. Having given that the centimetre is ·3937 inch and the gramme weighs ·0022046 lb., find the measures of the dyne and erg in foot-lb.-second units.

6. The unit force being the weight of a ton, the unit acceleration that due to gravity, the unit velocity that of a body which has fallen from rest 5 seconds, find the units of mass, length and time.

7. If the unit of mass is the mass of a ton, the unit of momentum that possessed by one lb. moving at the rate of one mile per hour, find the unit of velocity.

8. If one poundal is the unit of force and one foot-sec. the unit of velocity, show that there are as many lbs. in the unit of mass as there are seconds in the unit of time.

9. The mass of n lbs., n feet and n^2 seconds being taken as units, show that the unit force is one poundal.

10. The unit of work is that required to raise a ton weight through a vertical distance of 10 feet, the velocity of 10 miles an hour is the unit of velocity, find the unit of mass.

EXAMPLES. XXI.

1. Find the H.P. of an engine able to drive a train of 100 tons on a level line on which the resistance is $\frac{1}{160}$ of the load at a speed of 30 miles an hour.

2. A uniform india-rubber cord has a length of 27 inches under a tension of 4 lbs. weight and a length of 23 inches under a tension of 2 lbs. weight. Calculate the work done in stretching it from its natural length to a length of 30 inches.

3. The total resistance to a car with the brakes on is $\frac{1}{16}$ of the weight of the car; show that the car, when running at 8 miles per hour, can be stopped in about $11\frac{1}{3}$ yards. Show also that, if the ordinary resistance is $\frac{1}{64}$ of the weight, each stoppage by the brakes and recovery of the previous speed of 8 miles per hour adds to the work of traction an amount approximately equal to 181 foot-pounds per cwt. of the car mass.

4. A straight rod ACB without weight has two particles of equal weight fastened to it, one at the end B and the other at the middle point C, and the rod can swing about A. If it be held horizontally, and then allowed to swing, prove that the greatest velocity acquired by the end B will be the same as that of a particle which has fallen freely from rest through a height $=\frac{6}{5}$ of the length of the rod.

5. Prove that a train of W tons going up an incline of 1 in m will acquire a velocity $\left(\frac{P}{W}-\frac{1}{m}-\frac{R}{2240}\right)gt$, and energy

$$\frac{1}{2}W\left(\frac{R}{W}-\frac{1}{m}-\frac{R}{2240}\right)^2 gt^2,$$

foot-tons after t seconds from rest, P being the pull of the engine in tons, and R the resistance on the level in lbs. per ton.

6. Find the charge of powder required to send a 32 lb. shot to a range of 2500 yards with an elevation of 30°, supposing the initial velocity is 1600 feet per second when the charge is half the weight of the shot, and that the initial energy of the shot is always proportional to the charge of powder.

7. Having given that an engine of 60 H.P. is required to drive a steamer 80 feet long at the speed of 9 knots, find the H.P. of an engine which will drive a similar steamer 240 feet long and similarly immersed at 18 knots, assuming that the resistance is proportional to the wetted surface and to the square of the velocity through the water.

8. A fine string passes through two small fixed rings A and B in the same horizontal plane and carries equal weights at its ends. If a third equal weight is attached to the middle portion AB of the string and is let go prove that it will descend to a depth $=\frac{2}{3}AB$ below AB and then ascend again.

9. Find the H.P. transmitted by a belt moving with a velocity of 600 feet per minute passing round 2 pulleys, supposing the difference of tension of the two parts to be 1650 lbs.

10. An inelastic pile of ½ a ton is driven 12 feet into the ground by 30 blows of a hammer of 2 tons falling 30 feet. Prove that it would require 120 tons in addition to drive it down very slowly.

11. A train whose mass is m lbs. moves against a constant resistance equal to p times its weight; it starts from rest and moves with constant acceleration till the steam is shut off and arrives at the next station, distant a from the starting point in t seconds.

Show that the greatest horse-power exerted by the train is $C\dfrac{2mp^2g^2at}{pgt^2-2a}$, where C is a constant depending on the units employed.

12. Two equal masses, connected by a string 6 feet long, are lying close together on the ground. One of them is projected vertically upwards with a velocity of 50 feet per second; find the whole time that has elapsed when both are back again on the ground.

13. A, B, C are masses, weighing 5, 4 and 3 oz. respectively, which are fastened to an inelastic thread, passing over a smooth fixed horizontal cylinder, so that A hangs freely on one side, and B and C are on the other side, B hanging freely and C lying on a table vertically below B. At a given instant A and B begin to move, the string between B and C being slack. After two seconds the string becomes taut. How long will it be before C reaches the table again?

14. A wedge of mass $15m$ can slide freely on a smooth horizontal plane, and a particle of mass m slides down its smooth face which is inclined at an angle 45° to the horizon. If h be the initial height of the particle above the plane, show that when it reaches the plane, the wedge will have moved a horizontal distance $\frac{1}{16}h$.

15. A mass of 12 oz. is attached by two strings to two equal masses of 9 oz. The strings pass over two small smooth pulleys in the same horizontal straight line, distant 8 inches apart. The system is held in equilibrium with the two masses of 9 oz. hanging vertically and the mass of 12 oz. situated at the middle point of the distance between the pulleys. If the system is released, how far will the mass of 12 oz. descend before it begins to rise again?

16. A cyclist, who works at a uniform rate and whose mass together with that of his machine is 175 lbs., rides at the rate of 30 feet per sec. on the level and 20 feet per sec. up an incline of 1 in 100. That part of the force opposing the motion of the machine which is independent of the inclination of the road being assumed constant and equal to F, determine the constant velocity down an incline of 1 in 200, and prove that F is equal to 3½ lbs. weight.

CHAPTER V.

PARALLEL FORCES. MOMENTS.

78. THE motion of the bodies we have been considering so far has been merely one of *translation, i.e.* one in which all the points of the body move in parallel lines with the same velocity.

But when a force acts on a body it usually produces *rotation*.

For instance a billiard-ball on a smooth table if struck at a point begins to rotate as well as to move along the table. A blow applied to a smooth cube lying on a table renders this rotational movement still more obvious.

Thus the velocities of the different points of the body are not the same either in direction or magnitude.

In the case of a particle or very small body the motion of translation is the only one that need be considered.

Take the case of a body to which a single force P is applied. Owing to the cohesion of the particles of the body, each particle is acted upon by forces due to its connexion with the other particles, such forces are usually called *internal* forces.

For each particle the Second Law of Motion holds, viz. that *its mass-acceleration in any direction is equal to the force acting upon it in that direction.* Hence by addition we see that the total mass-acceleration of the body in any given direction is equal to the sum of all the forces acting on its particles in that direction.

Now it is easy to see that the internal forces destroy each other, for if A and B be two particles of the body, the force on A due to B is equal and opposite to the force on B due to A, by the Third Law of Motion, thus the internal forces taken together destroy each other in pairs.

We are therefore left with the force P, hence it follows that

the mass-acceleration of the body in any direction is equal to the component of P in that direction. If several forces Q, R, ... are applied in addition to P, we have, in like manner, that

the mass-acceleration of the body in any direction is equal to the sum of the components of P, Q, R ... in that direction

When a force acts at a point of a body the line through the point in the direction of the force, produced both ways, is called the **line of action** of the force. Thus in the figure the force represented by LM acting on the flat body of the shape represented in the figure at M, the line AB is called the line of action of the force LM.

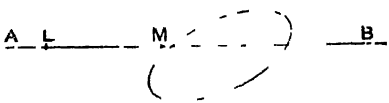

FIG. 51.

We shall show that the force LM produces the same effect at whatever point in its line of action it acts, provided the point is in the body.

79. Transmissibility of Force.

The body being acted on by a force P at the point C, let us apply a force at any point B in the line of action of P so as to keep B *fixed*.

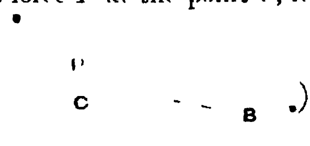

FIG. 52

That being done, the force P cannot turn the body about B, hence the body will not move, since B is *fixed*. Thus the body has no acceleration and therefore the force at B must, by what was shown in the last article, be equal and opposite to the force P.

Thus we see that two equal and opposite forces whose lines of action are the same produce exactly opposite effects on a body.

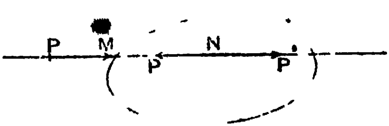

FIG. 53.

Now let a force P act at M, then at any point N in its line of action we can suppose to act two equal and opposite forces each of magnitude P, for two such forces produce no effect.

By what has just been shown the force at M and one of the forces at N destroy each other, and there remains a force P *acting at* N.

Thus we can replace a force by an equal force acting at any point in its line of action.

This fact is called the Principle of the Transmissibility of Force.

80. The Moment of a Force.

If a perpendicular be drawn from a point upon the line of action of a force, the product of the length of the perpendicular and the magnitude of the force is called the *moment of the force about the point.*

For instance, let AB be the line of action of a force whose

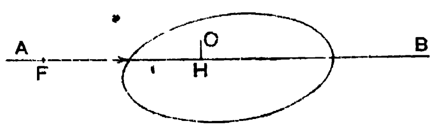

FIG. 54.

measure is F, from a point O draw OH perpendicular to AB. The moment of F about O is

$$Fp,$$

where p is the measure of OH; *e.g.* if F is 4 poundals, and OH is 3 feet the moment is 12.

81. The perpendicular vanishes if the line of action of the force passes through the point about which the moment is taken.

Hence the moment of a force about any point on its line of action is zero. Conversely, when the moment of a force about any point is zero, its line of action passes through that point.

It will be shown in Art. 86 that when a body under the action of a force turns round a point the moment of the force with regard to that point measures the *power* of the force to produce rotation.

The unit moment is that of one lb. wt. about a point one foot distant from it, and may be called one lb. ft. moment.

82. Sign of the Moment of a Force.

The moment of a force about a point is said to be *positive* when the force tends to turn the body in the direction opposite to that of the hands of a watch, and *negative* when in the same direction as the hands of a watch.

When there are several forces the sum of their moments about any point is, of course, their *algebraic* sum.

Ex. 1. $ABCD$ is a square whose side is 2 feet long; find the moments about B and C of the following forces; 4 lbs. along AB, 9 lbs. along CB, 2 lbs. along DA, and 20 lbs. along DC.

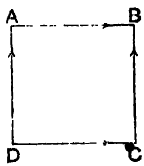

FIG. 55.

The moment about B of the force along
$\begin{cases} AB \text{ is zero,} \\ CB \text{ is zero,} \\ DC \text{ is } \quad 2\times 20 = \quad 40 \text{ units of moment,} \\ DA \text{ is } -2\times \ \ 2 = - \ \ 4 \quad \ldots\ldots\ldots\ldots\ldots \end{cases}$

The moment about C of the force along
$\begin{cases} AB \text{ is } -2\times 4 = -8 \text{ units of moment,} \\ CB \text{ is zero,} \\ DC \text{ is zero,} \\ DA \text{ is } -2\times 2 = -4 \quad \ldots\ldots\ldots\ldots\ldots \end{cases}$

Ex. 2. In the above square along the lines CB, BA, DA, DB forces act respectively equal to 4, 3, 2 and 5 lbs.; find the algebraic sum of the moments of the forces about C.

The moment about C of the force along
$\begin{cases} CB \text{ is zero,} \\ BA \text{ is } 6, \\ DA \text{ is } -4, \\ DB \text{ is } -5\sqrt{2}; \end{cases}$

hence the algebraic sum is $-5{\cdot}05$ units of moment, nearly.

83. Geometrical Representation of the Moment of a Force.

We have seen that if LM represents a force and OH is

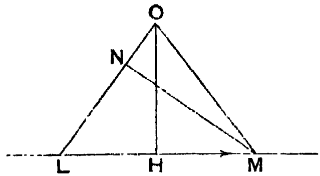

FIG. 56.

the perpendicular from O on the line of action of LM, then the moment of this force is measured by

$$OH \times LM,$$

but this product measures *twice the area of the triangle* OLM; hence the moment of a force about a point is measured by twice the area of the triangle formed by joining the point to the extremities of the line representing the force.

Again, twice the area of the triangle OLM is equal to

$$LO \times NM,$$

and NM is the component of LM perpendicular to LO, Art. 53.

84. The Moment of Resultant Force equals Sum of Moments of Component Forces.

Let AP and AQ be two forces having a resultant AR, we

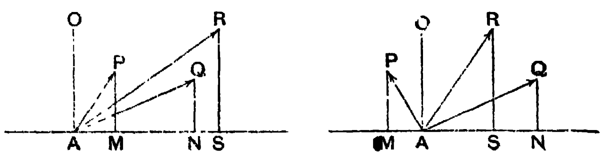

FIG. 57.

have to prove that the sum of the moments of AP and AQ about any point O is equal to the moment of AR about O.

Join AO, and draw $AMNS$ perpendicular to AO, draw also PM, QN, and RS parallel to AO.

The moment of AP about O is

$= AO \times$ component of AP perpendicular to AO, Art. 83

$= AO \times AM.$

Similarly,

the moment of AQ about O is $= AO \times AN$,

.... AR .... $= AO \times AS$,

$= AO \times (AN + AM)$, Art. 56

$=$ sum of moments of AP and AQ.

Notice that in the second figure the moment of AP is

$$-AO \, . \, AM.$$

85. It follows in the same manner as the foregoing, that when any number of forces act at a point, the moment of the resultant is equal to the sum of the moments of the component forces.

86. The Rotatory Power of a Force depends on its Moment.

Take a body moveable about a fixed point O, and let it be acted on by two forces P and Q whose moments about O are equal and opposite. It follows from what we have just seen that the moment of the resultant of P and Q is zero.

Hence this resultant passes through O, Art. 81, and therefore produces no rotation about O.

Thus the tendencies of P and Q to produce rotation are equal and opposite, and hence forces of *equal moment* have *equal rotatory powers.*

87. Moments of any Number of Forces.

When any number of forces *in one plane* have a resultant, the algebraic sum of their moments about any point O is equal to the moment of this resultant about O.

Let the forces be P, Q, R, ..., their final resultant is obtained by replacing,

P and Q by their resultant R_1,

R_1 and R .... R_2,

and so on until only one force is left.

Now the moment about O of R_1 = algeb. sum of the moments of P and Q, Art. 84,

.... R_2 = algeb. sum of the moments of R_1 and R,

= algeb. sum of the moments of P, Q, and R,

and so on.

Hence the moment of the final resultant = algeb. sum of the moments of all the forces P, Q, R, ...

88. Composition and Resolution of Parallel Forces.

We have shown how to find the resultant of any number of forces in one plane which act at a point.

If the forces are applied at different points of a body their resultant may be found, as in the last Article, by finding the resultant of any two intersecting forces, then the resultant of this and a third force, and so on.

In the case of parallel forces, which meet at an infinite distance, we adopt a special method for finding the resultant.

Like parallel forces are those which act in the same direction; *unlike* parallel forces are those which act in opposite directions.

89. Resultant of two like Parallel Forces.

Let P and Q be two parallel forces; we may suppose them to act at A and B respectively; at A and B introduce two forces F *equal and opposite,* this will make no difference in the resultant of P and Q, see Art. 79.

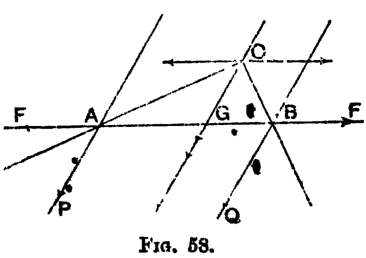

Fig. 58.

The lines of action of the resultants of F and P, and of F and Q will meet at some point O, and we may suppose these resultants to act at O. Art. 79.

Through O draw OG parallel to AP and BQ, cutting AB in G.

The force acting at O along OA may be replaced by its components, F parallel to GA, and P acting along OG.

Similarly the force acting at O along OB may be resolved into two: viz. F parallel to GB, and Q acting along OG.

The two forces F acting at O balance each other and may be removed. We are left with the single force

$$P + Q \text{ acting along } OG.$$

To determine the position of G.

The sides of the triangle OGA are parallel to P, F and their resultant, hence by the converse of the Triangle of Forces,

$$\frac{OG}{GA} = \frac{P}{F}.$$

Similarly
$$\frac{OG}{GB} = \frac{Q}{F}.$$

Hence
$$P \,.\, GA = Q \,.\, GB.$$

That is, *AB is divided in the inverse ratio of the forces.*

90. Resultant of two unlike Parallel Forces.

In this case P and Q are unlike parallel forces of which Q is the greater. Suppose them to act at any points A and B on their lines of action. At A and B apply equal and opposite forces F.

The resultants of F and P and of F and Q will meet in some point O which is *to the right of* Q, since, Q being greater than P, the resultant of Q and F is more bent towards Q than the resultant of P and F is bent towards P.

Through O draw OG parallel to AP and BQ and meeting AB in G.

The force acting at O along OA may be replaced by F parallel to GB, and P acting along OG.

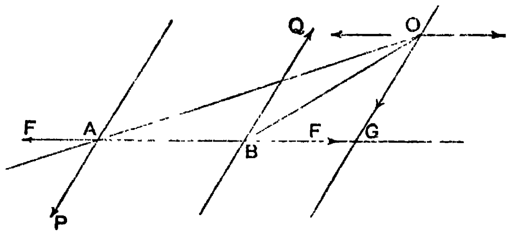

Fig. 59.

Similarly the force acting at O along OB may be replaced by F parallel to BG, and Q acting along GO.

The forces F balance each other, and we are left with

$$Q - P \text{ acting along } GO.$$

To determine the position of G.

The sides of the triangle OGA are parallel to P, F and their resultant, hence by the converse of the Triangle of Forces,

$$\frac{OG}{AG} = \frac{P}{F}.$$

Similarly $$\frac{OG}{BG} = \frac{Q}{F}.$$

Hence $$P \,.\, AG = Q \,.\, BG.$$

That is, *AB is divided externally in the inverse ratio of the forces.*

91. Recapitulation.

When two parallel forces P and Q act at points A and B of a rigid body;

1. The magnitude of the resultant is the sum or difference of P and Q according as they are like or unlike.

2. Its direction is that of each force when they are *like*, that of the greater when they are *unlike*.

3. It acts at a point G in AB, or in AB produced, such that

$$P.AG = Q.BG.$$

Cor. 1. The resultant of any number of parallel forces may be found by finding the resultant of any two of the forces, then the resultant of this and a third force, and so on.

The magnitude of this final resultant is the algebraic sum of the forces.

Cor. 2. The magnitude of the resultant of two equal and unlike parallel forces is zero, and it acts at an infinite distance. This case will be considered in Chapter V.

Ex. 1. Two like forces of 12 and 20 lbs. weight act at points 4 inches apart, find their resultant.

The resultant is a force of 32 lbs. weight.

Let x be the distance AG, then $BG = 4 - x$ and we have

$$12x = 20(4 - x),$$

$$\therefore\ 32x = 80, \quad \text{or} \quad x = 2\tfrac{1}{2} \text{ inches.}$$

Ex. 2. Two unlike forces of 48 and 72 lbs. weight act at points 14 feet apart, find their resultant.

The resultant is a force of 24 lbs. weight.

Let x feet be the distance BG, then $AG = x + 14$, and

$$48(x + 14) = 72x,$$

$$x = \frac{14 \times 48}{24} = 28 \text{ feet.}$$

Ex. 3. Resolve a force of 30 lbs. weight into two like forces 6 feet apart, one of them being 9 inches from the given force.

Let the forces be P and Q, then

$$P + Q = 30 \text{ lbs weight}, \quad AG = 9, \quad BG = 63,$$

also

$$9P = 63Q,$$

from which it follows that $P = 26\tfrac{1}{4}$ lbs., $Q = 3\tfrac{3}{4}$ lbs.

Ex. 4. Four forces F, $2F$, $3F$, $4F$ act along the sides of a square taken in order, find their resultant.

Let a be the length of a side. The resultant of F and $3F$ is $2F$ acting at a point E in BC produced such that

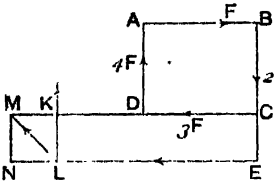

Fig. 60.

$$3F \cdot CE = F(a + CE),$$

or
$$CE = \frac{a}{2}.$$

Similarly the resultant of $2F$ and $4F$ is $2F$ acting at a point K in CD produced such that

$$4F \cdot DK = 2F(a + DK),$$

or
$$DK = a.$$

These two forces intersect in L, their resultant is $2\sqrt{2}F$ and acts along LM, the figure $LKMN$ being a square.

EXAMPLES. XXII.

1. Two parallel forces of 15 and 20 lbs. weight act at points 20 inches apart; find their resultant when they are (i) like, (ii) unlike.

2. The resultant of two like forces is 12 lbs. and it acts at a distance of 2 inches from the larger component which is 8 lbs.; find its distance from the smaller component.

3. The resultant of two unlike forces is 15 lbs. and it acts at distances of 2 feet and 6 feet respectively from the forces; find the forces.

4. Two men carry a weight of 160 lbs. between them on a pole, the weight being three times as far from one man as from the other; find how much weight each supports, the weight of the pole being disregarded.

5. A uniform rod 12 feet long and weighing 18 lbs., can turn freely about a point in its length; the rod is at rest when a weight of 8 lbs. is hung at one end. How far from the end is the point about which the rod can turn?

[The weight of the rod acts at its middle point.]

6. A bridge girder rests on two stone piers, the weight of the girder being a tons, and one of the piers being only capable of supporting b tons, at what distances must each pier be from the centre of the girder in order that the stronger pier may support as small a weight as possible?

7. A man carries a bundle at the end of a stick which is placed over his shoulder, if the distance between his hand and his shoulder be changed, how does the pressure on his shoulder change?

8. A uniform rod whose weight is 8 lbs. is placed upon two props which are in the same horizontal line and 6 inches apart. Find the distance to which the ends of the rod extend beyond the props, if the difference of the pressures on the props is 4 lbs., and the length of the rod 3 feet.

92. Moment of the Resultant of Parallel Forces.

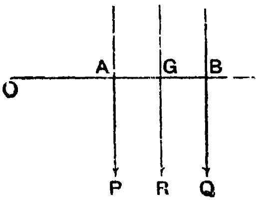

FIG. 61.

From O any point in the plane of two like parallel forces P and Q draw $OAGB$ perpendicular to the lines of action of the forces.

Now, sum of the moments of P and Q about O

$$=P.OA+Q.OB$$
$$=P(OG-AG)+Q(OG+GB)$$
$$=(P+Q)\,OG, \text{ since } P.AG=Q.GB, \text{ Arts. 89, 90,}$$
$$=\text{moment of the resultant about } O.$$

If the point O about which moments are taken is between the forces, we have

sum of moments of P and Q

$$=Q.OB-P.OA$$
$$=Q(OG+GB)-P(AG-OG)$$
$$=(P+Q)\,OG$$
$$=\text{moment of resultant.}$$

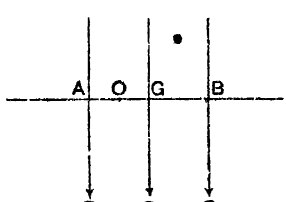

FIG. 62.

In precisely similar manner we show that when the parallel forces are unlike, the moment of the resultant about any point equals the sum of the moments of the components.

93. Referring to Art. 87, we see that the algebraic sum of the moments of any number of forces is equal to the moment of their resultant whether the forces meet at a finite distance or are parallel.

Ex. 1. Four weights of 5, 14, 6 and 25 lbs. respectively hang at distances 2, 4, 9 and 12 feet from one end of a rod without weight. Find the magnitude and the position of their resultant.

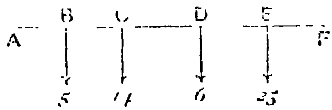

FIG. 63.

Let AF be the rod and B, C, D, E the points at which the weights are attached. Since the forces are like and parallel their resultant is their sum 50 lbs.

Let x be the distance of the resultant from A, then since

moment of resultant = sum of moments of components,

$$50 \cdot x = 5 \times 2 + 14 \times 4 + 6 \times 9 + 25 \times 12;$$

$$\therefore \quad x = \tfrac{42}{5} = 8\tfrac{2}{5} \text{ feet.}$$

Ex. 2. A rod 14 feet long without weight has a weight of 4 lbs. suspended from its middle point. The rod can turn about one end. If the rod is to be sustained by a force at one end of 11 lbs. weight, where must an additional weight of 63 lbs. be attached, in order that the rod may remain at rest?

The system of forces (including the force at the jointed end) must be in equilibrium, or there is no resultant force. Hence the algebraical sum of the moments about *any* point vanishes. Take moments about the jointed end. Then the force *at* that point has, of course, no moment and we have, if x is the distance from that end of the force of 63 lbs.,

$$63 \times x + 4 \times 7 - 11 \times 14 = 0;$$

$$\therefore \quad x = 2 \text{ feet.}$$

Ex. 3. A system of forces in one plane being represented in magnitude and position by the sides of a closed polygon taken in order, show that the sum of their moments with regard to any point O in the plane is *constant*.

(i) Let the point O be inside the polygon, then denoting the sides by a, b, c, &c. and the perpendiculars from O by p, q, r ... the sum of the moments is

$$pa + qb + rc + \ldots,$$

which is twice the area of the polygon.

(ii) Let the point be outside the polygon.

Here one of the forces has a moment about O opposite in sign to the moments of all the other forces, and

the sum of the moments $= pa - qb + rc + \dots$

$=$ twice area of polygon.

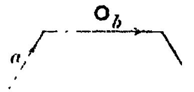

FIG. 64.

EXAMPLES. XXIII.

1. The resultant of two unlike parallel forces is 6 lbs., and acts 10 inches from the greater force which is 10 lbs.; find the distance between the forces.

2. A uniform bar 12 feet long and weighing 16 lbs. is supported at each end and a weight of 48 lbs. is hung at a point 2 feet from one end; find the pressure on each of the supports.

3. Four parallel forces 1, 6, 9, 8 act at points 4 inches apart along a weightless rod; where must the rod be supported that it may remain in equilibrium?

4. A uniform iron rod 6 feet long weighs 9 lbs., and from its extremities weights of 6 lbs. and 12 lbs. respectively are suspended. From what point must the rod be supported in order that it may remain balanced in a horizontal position?

[The weight of the rod acts at its middle point.]

5. A heavy uniform beam, whose mass is 50 lbs., is suspended in a horizontal position by two vertical strings each of which can sustain a tension of 35 lbs. without breaking. Where must a mass of 20 lbs. be placed so that one of the strings may just break?

6. A weightless rod has equal weights attached to it, one at 15 inches from one end and the other at 9 inches from the other; it is supported by two vertical strings attached to its ends; if each string cannot support a tension greater than the weight of 50 lbs., find the greatest magnitude of the equal weights.

94. Centre of Parallel Forces.

When any number of parallel forces act at fixed points of a body, we shall prove that there is a certain point, called the **centre** of the parallel forces, at which the resultant always acts, however the forces are turned round their points of application, provided they remain parallel to each other and of the same magnitude.

Let the parallel forces $P, Q, R\ldots$ act at points $A, B, C\ldots$. Join AB and divide it at g_1 so that

$$P.Ag_1 = Q.Bg_1.$$

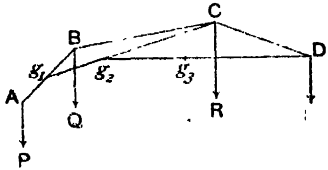

FIG. 65.

Then g_1 is a point at which the resultant of P and Q acts. Art. 89.

Now g_1 has been found independently of the *direction* of P and Q, hence it is the same whatever their direction.

Again join g_1 and C and divide it at g_2 so that

$$(P+Q)\,g_1g_2 = R.Cg_2.$$

Then g_2 is a point at which the resultant of $P+Q$ and R acts.

That is a point through which the resultant of P, Q and R passes.

Moreover we see that its position does not depend on the *direction* of the parallel forces.

Hence the resultant of P, Q and R passes through g_2, however these forces are turned round their points of application, provided they still remain parallel to each other. Thus g_2 is the *centre* of the parallel forces P, Q and R.

The centre of any number of parallel forces may be found by continuing this process.

Notice that the parallel forces are not restricted to lie in one plane.

95. Distance of the Centre from any Line.

To find the distance of the centre of any number of parallel forces P, Q, R ... from any line LM.

Since the position of the centre does not depend on the direction of the forces its position remains unaltered if we suppose the forces turned round so as to become parallel to LM.

From $A, B, C \ldots$ draw perpendiculars $p, q, r \ldots$ upon LM.

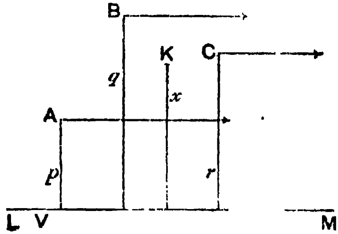

FIG. 66.

Let K be the required centre through which the resultant $P+Q+R \ldots$ passes, and let x be its distance from LM.

Then V being any point in LM, we have

moment of $P+Q+R \ldots$ acting at K about V = sum of moments of P, Q, ..., Art. 93;

$$\therefore \quad (P+Q+R+\ldots)\,x = Pp + Qq + Rr \ldots.$$

Hence, for instance, if there are only three forces

$$x = \frac{Pp+Qq+Rr}{P+Q+R}.$$

Ex. 1. Equal weights hang from the corners of a weightless triangle, find the point in the triangle at which it must be fastened in order to lie horizontally.

The resultant of any two of the forces acts midway between them, denoting each weight by P this resultant is $2P$.

Then if D is the middle point of BC, the resultant of $2P$ at D and P at A is $3P$ at G, where, GD being x and AD being l,

$$2P.x = P(l-x), \text{ or } x = \frac{l}{3}.$$

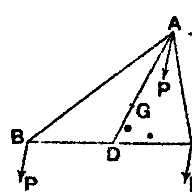

FIG. 67.

Ex. 2. A beam, the weight of which is equivalent to a force of 10 lbs. acting at its middle point, is supported on two props at its ends. If the length of the beam be five feet where must a weight of 30 lbs. be placed so that the pressure on the props may be 15 lbs. and 25 lbs. respectively?

The 30 lbs. weight is to be the resultant of the 10 lbs. weight and the upward pressures, hence if x be the distance of the 30 lbs. weight from the prop whose pressure is 15 lbs.

$$x = \frac{15 \times 0 + 25 \times 5 - 10 \times \frac{5}{2}}{30} = 3\tfrac{1}{3} \text{ feet.}$$

Ex. 3. Weights of 5, 6, 9 and 7 lbs. hang from the corners of a horizontal square whose side is 27 inches long. Find the point where a single vertical force must be applied to the square to balance the effects of the forces at the corners.

We want to find the centre of the four parallel forces; an upward force of 27 lbs. applied there will maintain equilibrium. The position of the centre is not altered if we turn the forces about their points of application till they have the position given in the figure.

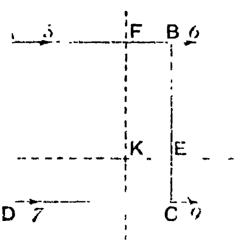

Fig. 68.

The resultant still passes through the centre of parallel forces; but its direction is the line KE, where

$$11 \,.\, BE = 16 \,.\, CE,$$

hence since

$$BC = 27 \text{ inches}, \quad BE = 16 \text{ inches}.$$

Similarly if the forces be turned till two of them lie upon DA and two upon CB their resultant acts along KF, and we find BF to be 12 inches. Thus K, the centre of parallel forces, is 16 and 12 inches respectively from two sides of the square.

96. Moments about a Line.

In Art. 80 we called the product of a force and its distance from a point O in its plane the *moment* of the force about the point. It is also the moment of the force about a line through O perpendicular to the plane of the paper.

In Art. 95 we found the relation

$$Pp + Qq + Rr + \ldots = (P + Q + R + \ldots)\,x;$$

where x is the distance of the centre of the parallel forces from the line LM.

Now turn all the parallel forces round their points of application till they are all perpendicular to the plane of the paper, then Pp, Qq &c. are the moments of the forces $P, Q, \ldots$ about the line LM; hence we see that the sum of the moments of the component forces $P, Q \ldots$ is equal to the moment of their resultant about LM.

EXAMPLES. XXIV.

1. A man and a boy have to carry a load of 100 lbs. slung on a pole (whose weight may be neglected) carried horizontally and 10 feet long. Their carrying powers are in the ratio of 8 : 5. Where in the pole should the weight be hung so that it may be fairly divided?

2. A rod of uniform thickness has half its length composed of one metal and the other half of another metal. The rod will balance about a point distant $\frac{1}{3}$ of its whole length from one extremity. Compare the weights of equal volumes of the two metals.

3. A uniform rod which is 12 feet long and which weighs 17 lbs. can turn freely about a point in its length, and the rod is in equilibrium when a weight of 7 lbs. is hung at one end. How far from the ends is the point about which it can turn?

4. Three like parallel forces acting at the angular points A, B, C of a plane triangle are respectively proportional to the opposite sides a, b, c. Find the distance of the centre of parallel forces from the side BC.

5. If the sum of the moments of a system of forces about a point A is zero and also about a point B, show that it is also zero about any point in AB.

6. Show that the difference of the moments of a force P about two points A and B in its plane equals the moment about either point of a force equal to P acting at the other point.

7. $ABCD$ is a rectangle, AB, BC adjacent sides are three and four feet long respectively. Along AB, BC, CD taken in order forces of 30, 40, 30 lbs. act respectively, find their resultant.

8. If any three forces act along the sides of a triangle taken in order, prove that their resultant cannot meet the triangle.

9. The lines of action of two forces P and Q and their resultant R are cut by a third line in the points A, B and C respectively, and P, Q are each resolved into two forces, one parallel to AB and the other parallel to R. Prove that the components parallel to R are to each other as $BC : AC$.

10. A uniform beam 4 feet long is supported in a horizontal position by two props which are three feet apart, so that the beam projects one foot beyond one of the props; show that the pressure on one prop is double the pressure on the other.

11. The sides BC, CA, AB of a triangle are three, four and five feet long respectively; find the magnitude and direction of a force acting at C whose moments about A and B are 7 and 5 respectively, and have opposite signs.

12. The magnitude of a force is known and also its moments about two given points A and B. Find by a geometrical construction its line of action.

13. A triangle ABC can turn freely in its own plane about the centre of its inscribed circle which is fixed, and forces proportional to

$$y-z,\ z-x,\ x-y$$

act along the sides BC, CA and AB respectively. Show that the triangle remains at rest.

14. Forces P, Q, R act along the sides BC, CA, AB of a triangle. Show that their resultant will act along the line joining the centre of the circumscribing circle to the intersection of perpendiculars if

$$P : Q : R = \frac{\cos B}{\cos C} - \frac{\cos C}{\cos B} : \frac{\cos C}{\cos A} - \frac{\cos A}{\cos C} : \frac{\cos A}{\cos B} - \frac{\cos B}{\cos A}.$$

15. Four forces acting along the sides AB, BC, CD and DA of the quadrilateral $ABCD$ are in equilibrium; having given that the first acts from A towards B, find the directions of each of the other three.

CHAPTER VI.

COUPLES.

97. We have already seen that if P and Q are two unlike parallel forces their resultant is equal to $P \sim Q$. When P is equal to Q the two forces are said to form a "couple." A couple therefore consists of two equal unlike parallel forces.

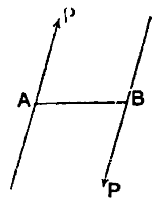

Fig. 69.

The perpendicular distance between the lines of action of the forces is called the "arm" of the couple.

98. Moment of a Couple.

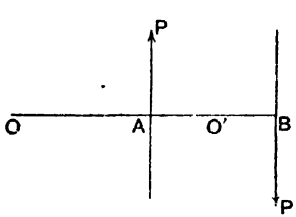

Fig. 70.

The forces P, P, whose arm is a, form a couple; it is required to find the sum of the moments of these forces about any point O.

Through O draw OAB perpendicular to the forces.

The sum of the moments equals

$$P\,.\,OB - P\,.\,OA, \text{ the moments having opposite signs,}$$
$$= P\,(OB - OA) = P\,.\,AB = P\,.\,a.$$

Again, if we take moments about any point O', within the forces, the sum of the moments equals

$$P\,.\,O'B + P\,.\,O'A,$$
$$= P\,(O'B + O'A) = P\,.\,AB = P\,.\,a.$$

Hence in all cases the moment of a couple about any point in its plane is equal to the product of one of the forces and the arm.

99. Sign of a Couple.

If a body on which the couple acts were pivoted about either the point O or the point O', the rotation would be in the direction of the hands of a watch. This is called the *negative* direction of rotation, see Art. 82, the contra-clockwise direction being called *positive*.

When the directions of the rotations produced by two couples are the same the couples are said to be *like*.

100. Axis of a Couple.

The axis of a couple is a line drawn through any point perpendicular to the plane of the couple of such magnitude as to indicate the magnitude of the couple.

The direction of rotation produced by the couple, or its *sign*, is indicated by the sign of its axis, which is determined as follows:

Place a watch on the plane of the couple face upwards; if the direction of rotation is contra-clockwise the axis is drawn *upwards* and is considered positive, if clockwise the axis is downwards and considered negative.

101. Couples with Equal and Parallel Axes of the same Sign are equivalent.

We shall now show that two couples in the same plane having equal moments, and therefore parallel and equal axes, are equivalent, that is, we *may replace a couple by any other couple in its plane having the same moment.*

First, take two couples of equal and opposite moment, the forces being all parallel.

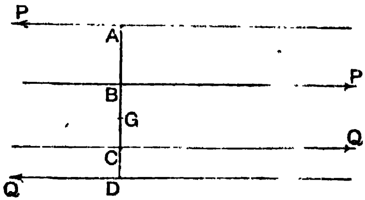

FIG. 71.

Draw a line $ABCD$ cutting the lines of action perpendicularly; then if $AB = a$, $CD = b$, and P, Q the forces of the respective couples, we are given that

$$Pa = Qb.$$

Observe that the moment of the upper couple is *positive*, that of the lower *negative*.

The resultant of the upper force P and the lower force Q is $P + Q$ acting at a point G which is such that

$$P(a + BG) = Q(b + CG), \text{ Art. 89.}$$

or
$$P \,.\, BG = Q \,.\, CG.$$

But the resultant of the two middle forces is $P + Q$ in the opposite direction and acts at the same point G.

Hence all the forces are in equilibrium since they can be replaced by two equal and opposite forces.

Thus the Q-couple destroys the effect of the P-couple, hence if the forces of the Q-couple were reversed they would be equivalent to the P-couple.

Second, if the forces of the couples are not all parallel, but form a parallelogram.

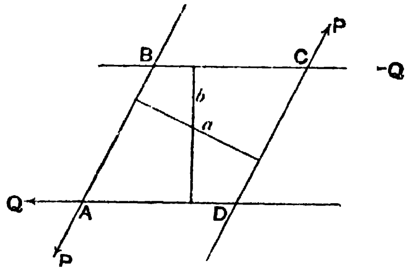

FIG. 72.

If a and b are the distances between the pairs of parallels, then since the moments are equal,

$$Pa = Qb,$$

but $$AB \,.\, a = AD \,.\, b,$$

since each is the area of the parallelogram $ABCD$, hence

$$\frac{P}{Q} = \frac{AB}{AD}.$$

Thus the sides of the parallelogram represent the forces P and Q in magnitude and direction, hence by Art. 43,

the resultant of the forces which meet at A is represented by CA,
....C.....................AC,

and these being equal and opposite forces the four forces are in equilibrium.

Thus if the forces of the Q-couple were reversed they would be equivalent to the P-couple.

102. The forces of a couple may be transferred to a parallel plane without altering their effect.

Take CD equal and parallel to AB and draw through it a plane parallel to that of the couple; at both C and D we may suppose equal and opposite forces of magnitude P to act in this plane.

Since AB and CD are equal and parallel they are

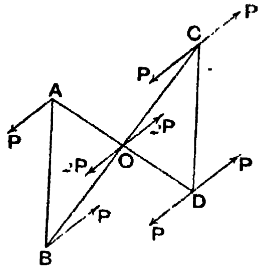

FIG. 73.

opposite sides of a parallelogram, hence AD and BC bisect each other at O.

The upward forces P at B and C have a resultant $2P$ acting upwards at O,

downward forces P at A and D have a resultant $2P$ acting downwards at O.

These forces $2P$ destroy each other and we are left with a downward force P at C and an upward force P at D, forming a couple exactly equal to the original couple.

103. Resultant of Couples in the same Plane.

Let P, Q, R, ... be the forces of any number of couples, their arms being p, q, r, ... respectively.

The moments of the respective couples are Pp, Qq, Rr, &c.

We may, by Art. 101, replace these couples by couples whose forces are

$$\frac{Pp}{L}, \frac{Qq}{L}, \frac{Rr}{L}, \dots$$

and whose arm is, in each case, L.

We may also, by the same Article, move the couples in their plane till their arms come to coincide, we have then one couple of which the force is

$$\frac{Pp}{L} + \frac{Qq}{L} + \frac{Rr}{L} + \dots,$$

and whose arm is L, that is, its moment is

$$Pp + Qq + Rr + \ldots,$$

the algebraic sum of the moments of the original couples.

Ex. 1. Forces P, $2P$, $4P$, $2P$ act along the sides of a square $ABCD$ taken in order, find the magnitude and position of the resultant.

The forces along BC and DA form a couple, this couple may be turned through a right angle, we then have forces $3P$ and $6P$ acting

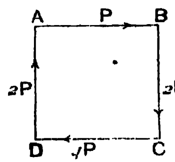

FIG. 74.

along AB and CD respectively; their resultant is $3P$ outside the side CD and distant from it a side of the square.

Ex. 2. Along the sides AB, CD of a square there act equal forces of 3 lbs. weight, along the sides AD and CB there act equal forces of 7 lbs. weight, find the moment of the resultant couple, a side of the square being 4 feet in length.

The moment of the couple formed by the last two forces is 28,
.... first .... −12;
hence the moment of the resultant couple is 16.

Ex. 3. $ABCD$ is a square whose side is 3 feet, along AB, BC, CD, DA forces act equal to 1, 3, 11, 7 lbs. weight respectively, and along AC, DB forces equal to $7\sqrt{2}$ and $3\sqrt{2}$ lbs. weight, find their resultant.

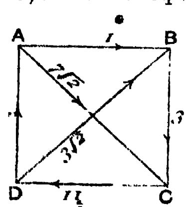

FIG. 75.

At the point B introduce forces of 3 lbs. weight acting along AB and BA.

By the Triangle of Forces, we may remove the forces acting along DB, BC, BA since they are in equilibrium.

There are left the forces along AC, CD, DA together with a force of 4 lbs. along AB.

At the point A introduce forces of 7 lbs. weight acting along AB and BA.

As before, the forces acting along AC, DA, BA may be removed.

There are left forces of 11 lbs. weight acting along AB and CD, these form a couple whose moment is 33 units of moment.

104. A Force may be replaced by a Force and a Couple.

Take any point O, then at O we may suppose two equal and opposite forces to act which are equal in magnitude to a given force P.

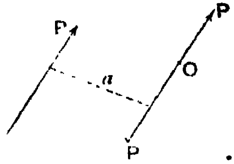

FIG. 76.

This given force together with one of the forces at O forms a couple, and there is left a force equal to P passing through O. Hence we have as equivalent to the original force P

(i) a force equal to P in magnitude and direction passing through O,

(ii) a couple whose moment is equal to Pa, where a is the perpendicular distance of the original force from O.

Conversely, we may replace a force and a couple by a single force.

For if the moment of the couple be Q and the force P, we may take the forces of the couple equal in magnitude to P, the arm being then $\frac{Q}{P}$.

Now let the forces of the couple be moved till one of them acts in the same line but opposite direction to the given force P thus balancing it, there is left one force equal to P and distant $\frac{Q}{P}$ from the given force.

Cor. A single force and a couple cannot produce equilibrium.

105. Any number of Forces in one Plane reduce to a Force or to a Couple.

Take any point O, then, as in the last Article, any force P may be replaced by an equal force through O together with a couple. Doing this for all the given forces we have

(i) a number of forces passing through O, which have a resultant force,

(ii) a number of couples which may be replaced by a resultant couple, Art. 103.

If the resultant in (i) does not vanish we are left with a force (through O) and a couple which as we have seen reduces to a single force.

If the resultant in (i) vanishes we are left with a couple.

EXAMPLES. XXV.

1. Forces 1, 2, 3, 4 act along the sides of a square taken in order; what forces must act in the diagonals that the whole system may be equivalent to a couple?

2. Prove that forces acting along, and represented by, the sides of any quadrilateral are equivalent to a couple, whose moment is represented by four times the area of the figure formed by joining the middle points of the sides.

3. Find the resultant of four forces of 3, 7, 11, 7 lbs. wt. acting along the sides of a square taken in order.

4. Forces of 5, 8, 8, 3 units act along the sides BA, BC, DC, DE of a regular hexagon $ABCDEF$; show that they are equivalent to a couple.

5. ABC is a triangle whose sides BC, CA, and AB are 25, 15 and 20 inches respectively. A force of 25 lbs. wt. acts along the bisector of the angle BAC; parallel forces through B and C are in equilibrium with this. If the forces are now turned about A, B and C respectively till they are at right angles to BC, prove that they form a couple of moment $42\frac{6}{7}$ inch-pounds.

6. Show that the resultant of forces represented by $m \,.\, OA$ and $n \,.\, OB$ is represented by $(m+n) \,.\, OC$, where C is a point on AB such that $m \,.\, AC = n \,.\, BC$.

7. Forces act along the sides AB, BC, CD, DA of a plane quadrilateral taken in order and their magnitudes are p, q, r, s times the lengths of the sides in which they act. Prove that they are equivalent to a couple if

$$(p-q)\,OB = (r-s)\,OD,$$

$$(q-r)\,OC = (s-p)\,OA,$$

where O is the intersection of AC and BD.

8. An equilateral triangle LMN has its angular points on the sides AB, CD, EF of a regular hexagon $ABCDEF$ whose centre is O. Show that the resultant of forces represented by LD, LE, MF, MA, NB and NC is a couple whose moment is represented by six times the difference of the triangles AOL and BOL.

CHAPTER VII.

CENTRE OF GRAVITY.

106. EVERY material body consists of an infinite number of particles, each particle being acted upon by a force, called its *weight*, directed to the centre of the Earth, due to the Earth's attraction. If the body be small compared with the Earth, these forces are practically parallel.

By the theory of like parallel forces, Art. 89, they have a resultant parallel to them and equal to the sum of the weights of the particles.

This resultant passes through the centre of the parallel forces, Art. 94, however the body be placed; in the present case this point is called the **Centre of Gravity.**

The centre of gravity of a body is therefore that point through which the line of action of the weight always passes, in whatever position the body may be.

107. Every Body has only one Centre of Gravity.

If possible let a body have two centres of gravity, A and B.

Then we have seen that the line of action of the weight passes through both A and B for every position of the body.

But this line of action is *vertical* and hence cannot pass through both A and B when the line AB is itself not vertical.

Hence there can only be one centre of gravity.

108. Position of the Centre of Gravity.

It has been shown that if there are any number of parallel forces P, Q, R ... acting at points whose distances

from a given line in their plane are p, q, r ..., the distance of the centre of parallel forces from that line is equal to

$$\frac{Pp + Qq + Rr + \ldots}{P + Q + R + \ldots}.$$

Now let there be any number of particles in a plane whose masses are m_1, m_2, m_3 and whose distances from a given line in that plane are z_1, z_2, z_3, ...

We have therefore a series of parallel forces m_1g, m_2g ....

The required distance is therefore

$$\frac{m_1gz_1 + m_2gz_2 + \ldots}{m_1g + m_2g + \ldots}, \text{ or } \frac{m_1z_1 + m_2z_2 + \ldots}{m_1 + m_2 + \ldots}.$$

The distance of the centre of gravity from any other line is obtained by a similar expression.

109. Since the C.G. is the point at which the body's weight may be supposed to act, the body if fixed at that point will balance about it in every position.

110. Determination of the Position of the C.G. by experiment.

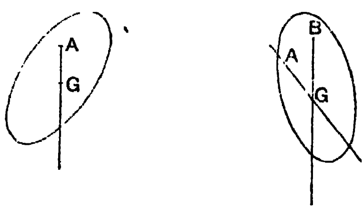

Fig. 77.

Suspend the body from any fixed point A in it.

The forces acting are its weight and the force at the point of suspension. Since the body is at rest these forces must be equal and opposite.

Therefore the vertical line through the C.G. must pass through A.

Release the body and suspend it from any other point B.

Then in this position also the C.G. must lie in BG.

Hence the C.G. will lie at G, the intersection of AG and BG.

111. Position of the C.G. found by Inspection.

If a body has a Centre of Symmetry, that point is its C.G.

By a Centre of Symmetry is meant a point O such that for every point P of the body we can find a point P' in PO produced so that $OP = OP'$; in the case of a circle the centre is evidently a centre of symmetry.

For in this case the body may be broken up into pairs of particles of equal weight on lines passing through the centre of symmetry at equal distances from it.

The centre of symmetry is then clearly the C.G. of each pair of particles and therefore of the whole body.

Hence it follows that:

the C.G. of a **uniform rod** is its **middle point,**

..............**circular ring** or **circular area** is its **centre,**

..............**sphere** is its **centre,**

..............**square** or **cube** is its **centre.**

112. C.G. of three equal Particles placed at the Vertices of a Triangle.

Let three equal particles, each of weight W, be placed at the vertices of a triangle ABC.

The C.G. of the weights at B and C is at D the middle point of BC, thus the C.G. of the three weights must lie in the median line AD and is a point G such that

$$2W \times DG = W \times AG, \text{ or } 2DG = AG.$$

Hence the point G divides AD in the ratio of 2 to 1, and

$$AG = \tfrac{2}{3}AD.$$

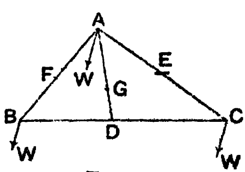

FIG. 78.

But we see in a precisely similar manner that the C.G. of the three weights must lie in the other two median lines

BE and CF and divide them in the ratio of 2 to 1. Hence the lines AD, BE and CF must all intersect in the point G which is the C.G. of the three weights.

113. C.G. of a Triangular Plate.

ABC is a thin triangular plate of uniform thickness.

Divide the plate into strips, such as PQ, parallel to the side BC. The C.G. of each strip is at its middle point, and the middle points of all these strips lie in the line AD joining A to D the middle point of BC. Hence the C.G. of the plate lies in AD.

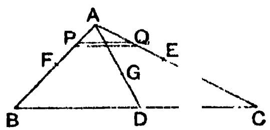

FIG. 79.

Similarly we see that the C.G. of the plate lies in BE and CF, hence by Art. 112 it is the same as the C.G. of three equal weights placed at the vertices of the triangle. Hence the C.G. divides each median line in the ratio of 1 to 2.

We have therefore proved that,

(i) The C.G. of a triangular area is the same as that of 3 equal particles placed at its vertices,

(ii) The C.G. of a triangular area is the point of intersection of the median lines and is distant from each vertex $\frac{2}{3}$ of the median, or

$$AG = \tfrac{2}{3} AD.$$

114. C.G. of a Parallelogram.

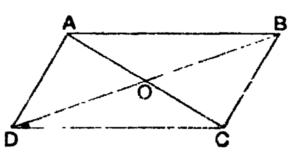

FIG. 80.

Since the diagonals of a parallelogram bisect each other, we see by the foregoing that the C.G. of the triangles ABC and ADC, that is of the whole figure, lies on BD, similarly it lies on AC. Hence it is at O, the intersection of the diagonals.

EXAMPLES. XXVI.

1. Show that the C.G. of a uniform rod is the same as that of equal particles at its ends.

2. Prove that a parallelogram has the same C.G. as four equal particles at its vertices.

3. Show, by dividing a parallelogram into strips parallel first to one pair of parallel sides and then to another, that its C.G. is its intersection of diagonals.

4. The C.G. of a triangle is the same as the C.G. of 3 equal particles placed at the middle points of its sides.

5. If the C.G. of a triangle coincides with the centre of the circumscribed circle, the triangle is equilateral.

6. A triangular board is suspended by a string attached to one corner. What point in the opposite side will be in line with the string?

7. Show that the C.G. of the circumference of a circle and that of any number of equal particles arranged at equal distances along its circumference are the same.

115. General Rule for finding the C.G.

Divide the body into portions whose weights and the positions of whose C.G.s are known.

The weight of each portion acts at its C.G., Art. 106. The C.G. may then be found.

116. When the body can be divided into two portions of which the weights w_1, w_2 and the C.G.s are known, we proceed as follows:

Let g_1, g_2 be the given centres, join g_1, g_2, then by the theory of parallel forces, Art. 89, the required point G divides g_1g_2, so that $w_1 . g_1G = w_2 . g_2G$.

Ex. 1. Find the C.G. of two spheres of 8 oz. and 24 oz. weights, connected by a rigid rod without weight, the distance between their centres being one foot.

Let x be the distance of the C.G. from the centre of the smaller sphere, then

$$8x = 24(12 - x),$$

$$\therefore \quad x = 9 \text{ inches.}$$

Ex. 2. A uniform rod weighing 7 lbs. is 6 feet long; if a 2 lb. weight be placed at one end, find the centre of gravity of the whole.

Since the rod is uniform its weight acts at its middle point. Let x be the distance of the C.G. from the middle of the rod, then

$$7x = 2(3 - x),$$

$$\therefore \; x = \tfrac{2}{3} \text{ feet, or 8 inches.}$$

Ex. 3. On the same base and on opposite sides of it isosceles triangles are described whose vertices are distant 12 and 18 inches respectively from the base. Find the C.G. of the quadrilateral.

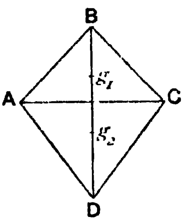

FIG. 81.

Let $ABCD$ be the quadrilateral: the line BD is perpendicular to, and bisects, the base AC. Hence g_1 and g_2, the C.G.s of the two triangles, are on BD.

The distance g_1g_2 is 10 inches, and the weights of the triangles are proportional to their areas.

Hence if x be the distance of the C.G. of the quadrilateral from g_2

$$\tfrac{1}{2}AC \times 18 \times x = \tfrac{1}{2}AC \times 12(10 - x),$$

or $$x = 4 \text{ inches.}$$

117. To find the C.G. of a Body when a portion is removed.

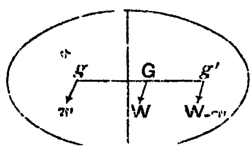

FIG. 82.

Let W be the weight of a body whose C.G. is G, w that of a portion of it whose C.G. is g, required the C.G. of the body got by removing the portion w from the body W.

Let x be the distance of its C.G. g' from G, then since W is made up of the portions w and $W - w$, we have

$$(W - w)x = w \, . \, gG, \quad \text{or} \quad x = gG\frac{w}{W - w}.$$

Ex. To find the C.G. of the figure $ABCD$ got by cutting the triangle COD out of the square $ABCD$ whose side is $\frac{3}{4}$ of an inch, O being the centre.

Let x be the distance of the required C.G. from O. Then we know that O is the C.G. of the whole square, and g that of COD, where $Og = \frac{2}{3}$ of half the side of the square $= \frac{1}{4}$ inch. Also area of square $= \frac{9}{16}$ sq. inch, area of $COD = \frac{9}{64}$ sq. inch.

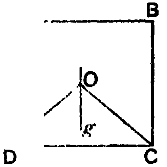

FIG. 83.

Hence since the weight of the square is made up of the weights of the fig. $ABCOD$ and of the triangle COD,

$$\left(\tfrac{9}{16} - \tfrac{9}{64}\right) x = \tfrac{1}{4} \times \tfrac{9}{64}, \text{ or } x = \tfrac{1}{12} \text{ inch.}$$

EXAMPLES. XXVII.

1. Find the C.G. of two small bodies whose weights are ·01 oz. and ·002 oz., their distance apart being 3 feet.

2. Two spheres whose radii are 10 and 11 inches respectively are in contact; if their weights are 5 and 6 lbs., find the position of their C.G.

3. Two uniform rods are placed so that the line 1 foot 10 inches long joining their centres is perpendicular to each, if their lengths are 10 and 12 inches find the distances of their C.G. from their ends.

4. A rod 12 inches long whose weight is 20 lbs. has a body of weight 2 ounces attached to a point one inch from an end, find the C.G. of the rod and attached weight.

5. Find the C.G.S. of the following bodies:

(i) A square with a square portion removed, the line joining their centres being perpendicular to a side of each. The sides of the squares are 10 and 3 inches long respectively, the distance between their centres being 2 inches.

(ii) A circular plate with a circular portion removed, the weights of the portions being 15 and 4 lbs. and the distance between their centres 15 inches.

(iii) A square plate with a circular portion removed, the boundary of this removed portion touching a side of the square and passing through its centre, the line joining their centres being perpendicular to a side of the square, the side of the square being 10 inches long.

(iv) A uniform rod with a piece $\frac{1}{4}$ of its length taken out, the centre of the piece being 3 inches from the centre of the rod.

118. C.G. of Weights in the same Straight Line.

Let the body whose C.G. is required be divided into any number of portions whose C.G.s lie in the same straight line,

let the weights acting at these points be w_1, w_2, ... and let their distances from some fixed point O on the line be x_1, x_2, ..., then the distance of the C.G. required from O being x, we have

$$(w_1 + w_2 + \ldots)\bar{x} = w_1x_1 + w_2x_2 + \ldots. \qquad \text{Art. 108.}$$

Ex. 1. Two heavy particles weighing respectively 3 and 5 ounces are attached to the ends of a straight rod 8 inches long, weighing 2 ounces. Find the C.G. of the system.

The sum of the weights is 10 ounces. Taking for the point O the end to which the 5 ounces are attached

$$10 \times \bar{x} = 3 \times 8 + 2 \times 4;$$

$$\therefore \bar{x} = 3\cdot2 \text{ inches.}$$

Ex. 2. A telescope consists of three tubes each 10 inches in length sliding within one another, and their weights are 8, 7 and 6 ounces. Find the position of the C.G. when the tubes are drawn out to their full length.

The sum of the weights is 21 ounces, the weights of the different tubes act at points distant 5, 15 and 25 inches respectively from one end.

$$\therefore 21 \times \bar{x} = 8 \times 5 + 7 \times 15 + 6 \times 25 = 295;$$

hence $\bar{x} = 14\frac{1}{21}$ inches from the thicker end.

EXAMPLES. XXVIII.

1. Ten 1 lb. weights are attached to points of a weightless rod distant one inch apart, find their C.G.

2. A rod is pivoted at its middle point and weights of 5 and 6 lbs. are attached to its ends, the rod being 20 inches long, where must a weight of 3 lbs. be attached in order that the rod may rest horizontally?

3. Four triangles have the same base and the opposite vertices in the same line perpendicular to the base, the distances of these vertices from the base being 5, 6, 7 and 8 inches, find the distance of the C.G. of the four triangles from the base.

4. A uniform rod AB is 6 feet long and weighs 4 lbs. A lb. weight is attached to the rod at A, 2 lbs. at a point distant one foot from A, 3 lbs. at 2 feet from A, 4 lbs. at 3 feet from A and 5 lbs. at B. Find the distance of the C.G. of the system from A.

5. A straight rod 6 feet long and heavier towards one end is found to balance about a point 2 feet from the heavier end, but when sup-

ported at its middle point it requires a weight of 3 lbs. to be hung at the lighter end in order to keep it level. What is the weight of the rod?

6. A bar of uniform thickness and 5 lbs. weight has a weight of 10 lbs. at one end and 12 lbs. at the other, it balances about a point 4 inches from the nearer end, find its length.

119. C.G. found by taking Moments about a Line.

If there are several bodies of weights $w_1, w_2, \ldots$ and if $x_1, x_2, \ldots$ are the distances of their C.G.s from any line Ox in their plane we have, by Art. 95, if $\bar{x}$ is the distance of their C.G. from this line

$$\bar{x} = \frac{w_1 x_1 + w_2 x_2 + \ldots}{w_1 + w_2 + w_3 + \ldots}.$$

Similarly if Oy be any other line, and $y_1, y_2, \ldots$ the distances of the C.G.s from it, then $\bar{y}$ being the distance of the C.G. from this line

$$\bar{y} = \frac{w_1 y_1 + w_2 y_2 + w_3 y_3 + \ldots}{w_1 + w_2 + w_3 + \ldots}.$$

Ex. 1. Four heavy particles whose masses are 2, 3, 4 and 5 grammes are placed at the corners A, B, C, D of a horizontal square; find the C.G. of the four particles.

Taking moments about AD, a being the length of a side,

$$(2+3+4+5)\,GM = (3+4)\,a,$$

$$\therefore\ GM = \tfrac{7}{14}\,a = \frac{a}{2}.$$

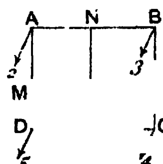

Fig. 84.

Similarly, taking moments about AB,

$$(2+3+4+5)\,GN = (4+5)\,a,$$

$$\therefore\ GN = \tfrac{9}{14}\,a.$$

Ex. 2. Five masses of 1, 2, 3, 4 and 5 ounces weight respectively are placed on a square table. Their distances from one edge of the table are 2, 4, 6, 8 and 10 inches, from an adjacent edge 3, 5, 7, 9 and 11 inches. Find the distance of their C.G. from these two edges.

One distance $\bar{x}$ is given by

$$\bar{x} = \frac{1\times2+2\times4+3\times6+4\times8+5\times10}{1+2+3+4+5},$$

$$= 7\tfrac{1}{3} \text{ inches}.$$

Similarly $\bar{y} = 8\tfrac{1}{3}$ inches.

Ex. 3. Weights proportional to 3, 4 and 5 are placed at the corners of an equilateral triangle whose side is of length a, find the distance of their C.G. from the first weight.

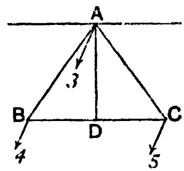

FIG. 85.

Take moments about a line through A parallel to BC; then since the weight at A has no moment about this line, we get

$$\bar{x}=\frac{4\times\frac{\sqrt{3}}{2}a+5\times\frac{\sqrt{3}}{2}a}{12}=\frac{3\sqrt{3}}{8}a.$$

Similarly taking moments about AD, the perpendicular to BC,

$$\bar{y}=\frac{5\times\frac{a}{2}-4\times\frac{a}{2}}{12}=\frac{a}{24}.$$

$$\therefore\quad AG=\sqrt{\bar{x}^2+\bar{y}^2}=\frac{\sqrt{(9\sqrt{3})^2+1}}{24}a=\cdot 6a \text{ nearly.}$$

Ex. 4. To find the C.G. of the perimeter of a triangle formed by three rods of uniform section.

Let D, E, F be the middle points of the sides of the triangle ABC, then p, q, r being the perpendiculars from A, B and C on the opposite sides, we know that

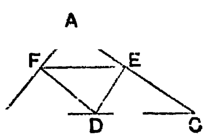

FIG. 86.

(i) EF, FD and DE are parallel to BC, CA and AB respectively,

(ii) the perpendiculars from D, E, F in EF, FD and DE are $\frac{1}{2}p$, $\frac{1}{2}q$ and $\frac{1}{2}r$ respectively.

We may suppose the weight of each rod to act at its middle point, hence taking moments about EF, we get as the distance of the C.G. from EF,

$$\frac{BC\times\frac{p}{2}}{BC+CA+AB}.$$

But $\frac{1}{2}p \times BC$ = area of triangle $ABC = S$; hence the distance of the C.G. from EF equals $\frac{S}{BC+CA+AB}$.

In the same way it may be shown that the distances of the C.G. from the sides ED and DF may also be shown to equal $\frac{S}{BC+CA+AB}$; hence since its distances from the sides of the triangle DEF are all equal it is the centre of its inscribed circle.

EXAMPLES. XXIX.

1. Weights of 2, 3, 4 and 5 lbs. respectively are placed at the corners of a square, and weights of 1, 6, 7, 8 lbs. are placed between them, viz. the weight 1 halfway between the weights of 2 and 3 lbs., the weight of 6 lbs. halfway between the weights of 3 and 4 lbs., and so on. Find the C.G. of all the weights.

2. Find the distance of the C.G. of half a hexagon from its base.

3. Three equal uniform rods are placed so as to form three sides of a square; find their C.G.

4. Equal particles are placed at 5 of the vertices of a regular hexagon, find the position of their C.G.

5. Weights of 1, 2, 3, 4, 5 and 6 lbs. are placed at the vertices of a regular hexagon, find their C.G.

6. Squares are described on the three sides of an isosceles right-angled triangle, find the C.G. of the complete figure so formed.

7. Four books are placed one above another on a table. The lowest projects 4 inches over the edge of the table, the next 2 inches beyond the lowest, the next $\frac{1}{2}$ an inch beyond the second, the uppermost 1 inch beyond the third. Each book is 16 inches in breadth and length and has an edge parallel to the edge of the table. Find the distance of the C.G. of the 4 books from the edge.

8. Find the C.G. of a figure in the shape of a cubical box without a lid, the sides being of small uniform thickness and an edge of the box being 14 inches long.

120. ***The work done by the weights of the system of particles forming a body in any displacement is equal to the work done by their resultant acting at their C.G.***

Let $w_1, w_2, w_3, \ldots$ be the weights of the particles,

$h_1, h_2, h_3, \ldots$ be their distances from any horizontal plane, also let

$x_1, x_2, x_3, \ldots$ be the displacements.

Then if $\bar{h}$ be the height of the C.G. before displacement, and $\bar{h} + \bar{x}$ after, we have

$$w_1 h_1 + w_2 h_2 + \ldots = (w_1 + w_2 + \ldots)\bar{h}, \qquad \text{Art. 96};$$

also

$$w_1(h_1 + x_1) + w_2(h_2 + x_2) + \ldots = (w_1 + w_2 + \ldots)(\bar{h} + \bar{x}),$$

hence subtracting,

$$w_1 x_1 + w_2 x_2 + \ldots = (w_1 + w_2 + \ldots)\bar{x};$$

which proves the theorem.

121. C.G. of the Tetrahedron.

Divide the tetrahedron into indefinitely thin plates parallel to the base BCD, PQR being one of them.

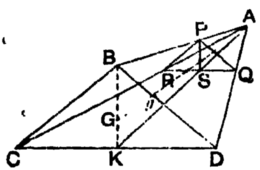

FIG. 87.

The C.G. of the triangular plate PQR lies in the line joining P to S the middle point of QR, Art. 113.

The points such as P for the different plates all lie in the line AB, the points such as S lie in the line AK, where K is the middle point of CD. Art. 113.

Hence the plane through AB and AK contains the C.G. of all the plates and hence of the whole tetrahedron.

In a similar way we may show that the C.G. of the tetrahedron lies in all the planes which pass through an edge of the tetrahedron and the middle point of the opposite edge; it is therefore the point of intersection of these planes.

Again, suppose four equal weights placed at A, B, C and D, the C.G. of the weights at C and D is at K, hence the C.G. of the four weights lies in the plane through AB and AK.

Similarly it is seen to lie in any plane through an edge and the middle point of the opposite edge.

Thus the tetrahedron and the four equal weights have the same C.G.

But we have seen, Art. 112, that the C.G. of three equal

weights at B, C and D is at G the C.G. of the triangle BCD, hence the C.G. of the four weights, and also of the tetrahedron, is at g, where

$$Ag = 3gG,$$

or it lies $\frac{3}{4}$ of the way down the line joining a vertex to the C.G. of the opposite face.

122. C.G. of a Pyramid standing on any Plane Polygon.

A pyramid whose vertex is O stands on the base formed by the polygon $ABCDEF$. Let K be the C.G. of the base, and join K to O and to the corners of the base.

The C.G. of the pyramid is the C.G. of the tetrahedra

$$OKAB,\ OKBC,\ \ldots\ldots$$

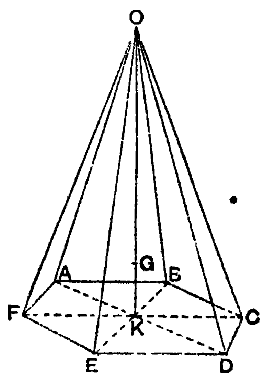

FIG. 88.

Take a section $A'B'C'D'E'F'$ of the pyramid by a plane parallel to the base at a distance from O of $\frac{3}{4}$ of the height of the pyramid; let OK cut this plane in G. Now the C.G. of each tetrahedron lies in this plane of section at the C.G. of the section of each by the plane (§ 121); also since the tetrahedra have the same height their masses are proportional to their bases and hence to the triangles $GA'B'$, $GB'C'$, ...

Consequently the C.G. of the pyramid lies at the C.G. of the section $A'B'C'D'E'F'$; that is, at the point G, since the line OK cuts all sections parallel to the base at the C.G. of the section.

Hence the C.G. of the pyramid is $\frac{3}{4}$ of the way down the line joining the vertex to the C.G. of the base.

123. C.G. of any Pyramid.

We may suppose the number of sides of the polygon in

the previous article to increase indefinitely, thus giving a curvilinear base. The result then becomes:—

The C.G. of a pyramid whose base is any closed curve is $\frac{3}{4}$ of the way down the line joining the vertex to the C.G. of the base.

124. The Arc of a Circle.

Let BAC be a uniform circular arc of mass m per unit length, of radius a and let the angle BOC be 2α in circular measure.

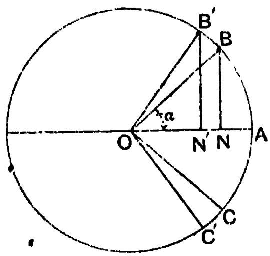

FIG. 89.

The C.G. clearly lies on the middle radius OA; let its distance from O be x. Then the moment M of the arc about O is given by

$$M = (\text{mass of arc})\, x = 2ma\alpha x \ldots(1).$$

Now suppose the angle α of the arc increased by a *small* amount BOB' as in the figure, so that the arc is now $B'AC'$; consider the increase which is made by this in the moment M. We have, since the changes are all small,

Small increase in M

$$= (\text{mass added}) \times (\text{distance of its C.G. from } O)$$

$$= (2m \,.\, \widehat{BB'}) \times (a \cos \alpha)$$

$$= 2ma \widehat{BB'} \cos \alpha = 2ma\,(B'N' - BN)$$

$$= 2ma \times (\text{small increase in } a \sin \alpha).$$

Hence $$M = 2ma^2 \sin \alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

M being zero when α is zero.

Therefore we have, from (1),

$$x = \frac{a \sin \alpha}{\alpha} = \text{radius} \times \frac{\text{chord}}{\text{arc}}.$$

125. Sector of a Circle.

Let the sector be divided into an indefinitely great number of small equal sectors. Each of these small sectors may be regarded as a triangle, whose C.G. therefore lies $\frac{2}{3}$ of the way down the radius.

The C.G.s of all the small sectors lie on an arc of a circle whose radius is $\frac{2}{3}OA$, and the mass of the whole sector may be supposed uniformly distributed over this arc.

Hence if G is the C.G. of the sector

$$OG = \tfrac{2}{3} a \times \frac{\sin\alpha}{\alpha} = \tfrac{2}{3}\frac{OA \times \text{chord } AB}{\text{arc } AB}.$$

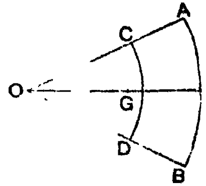

FIG. 90.

126. Zone of a Sphere.

Let the hemisphere ABC be cut by two planes parallel to its base, they intercept between them a portion called a zone. The planes also intercept between them a zone on a circular cylinder whose base is the same as that of the hemisphere. We shall show that these two zones have the same area and the same C.G.

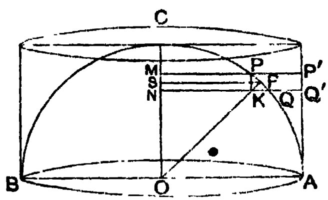

FIG. 91.

For take two planes very near together, they will intersect on the sphere and cylinder two elementary zones or belts. One elementary zone is produced by the revolution of PQ about OC and the other by the revolution of $P'Q'$. Let R be the middle point of PQ, join OR and draw RS parallel to OA. The arc PQ may be regarded as straight, and since the triangles PKQ and OSR are similar we have

$$\frac{PQ}{PK} = \frac{OR}{RS} \quad \text{.................................(i).}$$

Now the belt on the sphere may be regarded as a truncated cone, and the area of its surface is therefore equal to

the slant side $\times$ mean of bounding perimeters,

$$= PQ \times 2\pi RS$$
$$= 2\pi OR \times PK, \text{ from (i)}$$
$$= 2\pi OA \times P'Q'$$
$$= \text{area of belt on cylinder.}$$

Hence the areas of the belts on the sphere and cylinder are equal.

Thus the corresponding elementary zones on the sphere and cylinder have the same areas, they have also the same C.G., viz. S.

To find the C.G. of the entire spherical belt we suppose the weights of the elementary zones to act at their C.G.s in the line OC, and then find the C.G. of these weights.

But we have just seen that we get the *same weights at the same points*, by finding the C.G. of the cylindrical belt.

Hence the entire belts on the sphere and cylinder have the same C.G. But the C.G. of the cylindrical belt is halfway between the bounding planes, hence the C.G. of the spherical belt is also halfway between the bounding planes.

For a hemisphere, therefore, the C.G. is halfway down the radius.

127. Sector of a Sphere.

Let the given sector be divided into an indefinitely greater number of equal pyramids, by dividing the spherical surface into small equal areas and joining their boundaries to O.

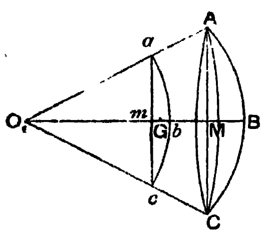

FIG. 92.

The C.G. of each of those pyramids lies $\frac{3}{4}$ of the way down the corresponding radius, Art. 123.

Hence the C.G.s all lie on a spherical cap abc and the mass of the sector may be supposed to be uniformly distributed over this cap.

But we have seen that the C.G. of such a cap is halfway between m and b, hence if G is the required C.G.

$$OG = Om + \tfrac{1}{2}mb$$
$$= \tfrac{3}{4}OM + \tfrac{3}{8}MB$$
$$= \tfrac{3}{8}(OM + OB).$$

For a hemisphere OM vanishes, hence $OG = \frac{3}{8}OB$.

EXAMPLES. XXX.

1. A square of cardboard is divided into four equal squares, one of the squares being cut out, find the C.G. of the remainder.

2. Four equal heavy particles lie in a straight line $ABCD$. If their mutual distances are a, ar, ar^2 respectively, find r when the C.G. is at C.

3. At each angle of a square and at the middle points of the sides are placed weights 1 lb., 2 lbs., 3 lbs., 4 lbs., 5 lbs., 6 lbs., 7 lbs. and 8 lbs., beginning at an angular point and going on in order round the perimeter, and a weight of 8 lbs. is placed at the intersection of the diagonals. Find their C.G.

4. Weights 5, 4, 6, 2, 7, 3 are placed at the corners of a regular hexagon taken in order, find their C. G.

5. ABC is a triangle, find a point O in it such that forces represented by OA, OB and OC shall be in equilibrium.

6. If the C. G. of a quadrilateral coincides with that of four equal weights at the vertices of the quadrilateral, show that the quadrilateral is a parallelogram.

7. A uniform rod is broken into two parts of five and seven inches length which are then placed so as to form the letter T, the longer portion being vertical, find the position of the C. G.

8. A square is bisected by a line through its centre making a given angle with the sides; find the C. G. of either half.

9. If weights 1, 2, 3, 4, 5 and 6 are situated at the angles of a regular hexagon the distance of their C. G. from the centre of the circumscribing circle is $\frac{2}{7}$ of the radius.

10. Five masses of 1, 2, 3, 4, 5 ounces weight respectively are placed on a square table. Their distances from one edge of the table are 2, 4, 8, 8, 10 inches and from the adjacent edge 3, 5, 7, 9, 12 inches. Find the distance of their C. G. from the two edges.

11. From a body of weight W a piece of weight w is cut off and moved a distance x; show that the C. G. of the whole is thereby moved a distance $\frac{xw}{W}$ in that direction.

12. Prove that the C. G. of four equal weights at the angles of a quadrilateral coincides with the C. G. of the parallelogram formed by joining the middle points of the sides.

13. The angle B of a triangle ABC is a right angle, AB is $7\frac{1}{2}$ inches and BC is 12 inches, at A, B and C are placed particles whose weights are proportional to 4, 5 and 6 respectively, find the distance of their C. G. from B.

14. Three forces PA, PB, PC diverge from the point P and three others AQ, BQ, CQ converge to the point Q; show that the resultant of the six is represented in magnitude and direction by $3PQ$.

15. A rod 12 feet long has a weight of 1 lb. suspended from one end and when 15 lbs. are suspended from the other end it balances at a point 3 feet from that end, while if 8 lbs. are suspended there it balances at a point 4 feet from that end. Find the weight of the rod and the position of its C. G.

16. A uniform metallic plate is cut in the form of a quadrilateral $ABCD$ so that the diagonal AC bisects it and BD cuts it into two parts in the ratio of 2 : 1. Show that its C.G. divides AC in the ratio of 5 : 4.

17. Two bodies are projected together from a point with different velocities and elevations. Show that their C.G. moves as if it were a heavy particle projected from the same point.

18. A triangular plate ABC, obtuse angled at C, stands on a table in a vertical plane having the side AC in contact with the table, show that the least weight which suspended from B will overturn it is

$$\tfrac{1}{3} W \frac{a^2+3b^2-c^2}{c^2-a^2-b^2}.$$

19. If one diagonal of a quadrilateral bisects the other and the C.G. be at the middle point of the bisecting diagonal, show that the quadrilateral is a parallelogram.

20. Show that the C.G. of a trapezium $ABCD$, in which AB is parallel to CD, divides the perpendicular distance between the parallel sides in the ratio $AB+2CD : 2AB+CD$.

21. Two sides of a rectangle are double of the other two, and on the longer side an equilateral triangle is described; find the C.G. of the figure made up of the rectangle and triangle.

22. A circle of radius r touches internally at a fixed point a fixed circle of radius R; find the distance of the C.G. of the area between them from their common tangent, and its ultimate position when r increases and becomes ultimately equal to R.

23. At the bottom of a coal-mine 275 feet deep there is an iron cage containing coal weighing 14 cwt., the cage itself weighing 4 cwt. 109 lbs. and the wire rope that raised it 6 lbs. per yard. Find the work done when the load has been raised to the surface and the H.P. required to do that amount of work in 40 seconds.

24. A cylindrical shaft has to be sunk to a depth of 100 fathoms through chalk, the weight of a cubic foot being 143·75 lbs., the diameter of shaft 10 feet, what H.P. is required to lift out the material in 12 working days of 8 hours each?

25. A tank 24 feet long, 12 feet broad, 16 feet deep is to be filled with water from a well the surface of which is always at a depth of 80 feet below the bottom of the tank, find the work done in filling the tank and the H.P. of an engine that will fill the tank in 4 hours, a cubic foot of water weighing 62·5 lbs.

26. From a triangular lamina ABC a triangular piece $A'B'C'$ is cut out, A' being the middle point of BC, and $B'C'$ being parallel to BC and equal to $\frac{1}{3}BC$. Show that the C.G. of the remainder is on AA' at a distance $\frac{44}{63}AA'$ from A.

27. A hollow is scooped out of a homogeneous hemisphere in the shape of a right cone on the same base. If the vertex of the cone coincides with the centre of mass of the remaining solid, how much has been removed?

28. A right circular cone, whose vertex is V and vertical angle $\frac{1}{3}\pi$, is inscribed in a sphere whose centre is C. If G be the C.G. of matter of uniform density filling the whole space between the cone and the sphere, prove that

$$VG : VC = 175 : 184.$$

29. A closed vessel is formed by a thin right circular cylinder, one end of which is closed by a hemispherical cap and the other by a flat plate, the whole being of uniform thickness. Find the ratio of the height of the cylinder to its radius that the vessel may rest in neutral equilibrium anywhere on its curved surface.

CHAPTER VIII.

FORCES IN EQUILIBRIUM.

128. WHEN forces act on a body in such a manner as to produce no motion they are said to be in *equilibrium.*

The simplest case is that of two equal and opposite forces having the same line of action.

129. Equilibrium of Three Forces in One Plane.

When three forces in one plane are in equilibrium their lines of action meet in one point.

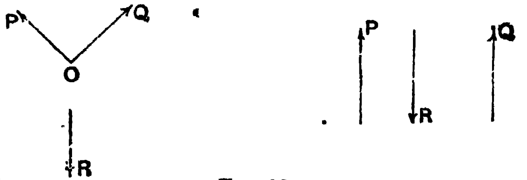

FIG. 98.

For the resultant of two of the forces P and Q may be supposed to act at the intersection O of their lines of action, but this resultant must have the same line of action as the third force R, since there is equilibrium, hence the line of action of R must pass through O.

In the case when the forces P and Q are parallel their resultant is parallel to them and hence also R must be parallel to P and Q.

Ex. 1. A uniform rod AB is suspended with its end in contact with a smooth vertical wall AC, by a string CE. If AE is $=\frac{1}{2}AB$, show that AB will be horizontal.

Since the rod is uniform the weight acts at its middle point of AB.

The forces acting on the rod are, its weight, the tension of the string, and the reaction of the wall (which is perpendicular to the wall).

Two of these forces, viz. the weight and the tension of the string, pass through E, hence also the reaction of the wall must pass through E, or AE must be perpendicular to the wall, *i.e.* horizontal.

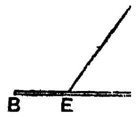

FIG. 94.

Ex. 2. A rod AB can turn round a fixed hinge at A and the end B rests against a smooth vertical wall. Find the reaction of the wall and the reaction of the hinge, the weight of the rod being 10 lbs., its length 18 inches, and the distance of the hinge from the wall 6 inches.

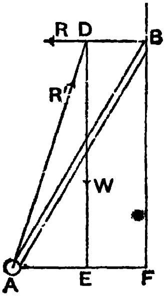

FIG. 95.

The weight of the rod acts vertically through the middle point of AB, the reaction R of the wall acts along BD perpendicular to the wall, let D be the intersection of the lines of action of these forces. We see that R', the reaction of the hinge, must pass through D. The sides of the triangle ADE are proportional to the forces. Art. 50.

Hence $$\frac{R}{W}=\frac{EA}{DE}, \quad \frac{R'}{W}=\frac{AD}{DE}.$$

$$EA=\tfrac{1}{2}AF=3;\ DE=BF=\sqrt{AB^2-AF^2}=\sqrt{18^2-6^2}=12\sqrt{2};$$
$$AD=\sqrt{9+(12\sqrt{2})^2}=\sqrt{297}.$$

Thus $$R=10\times\frac{3}{12\sqrt{2}}=10\times\frac{\sqrt{2}}{8}=1\cdot8 \text{ lbs. weight nearly.}$$

$$R'=10\times\frac{\sqrt{297}}{12\sqrt{2}}=10\cdot3 \text{ lbs. weight nearly.}$$

130. Stable, Unstable, and Neutral Equilibrium.

When a body is slightly displaced from its position of rest the forces which in its new position act upon it may restore it to its former position or may remove it still further from that position.

In the former case the equilibrium is called **stable,** in the latter **unstable.**

Instances of stable equilibrium are, a cube resting with a face upon a horizontal plane, or a weight hanging by a string.

A case of unstable equilibrium is in a "top-heavy" body, or a cube balanced on one of its edges.

When the body remains at rest in its *new* position the equilibrium is called **Neutral.**

Instances are a sphere, or a cylinder with its axis horizontal, on a horizontal plane.

All the kinds of equilibrium are exemplified in the following cases:

(i) an egg standing on its end is in unstable equilibrium,

(ii) lying on its side for displacements in *one* direction its equilibrium is stable, in *another* neutral.

131. Case of a Heavy Body with one Point fixed.

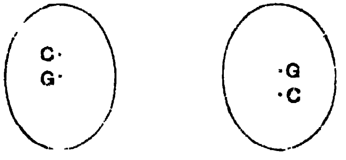

FIG. 96.

When a heavy body has one point fixed, since there are only two forces acting upon it, viz. its *weight* and the *force at the fixed point*, for equilibrium we must have these two forces acting in the same line, which must therefore be vertical, since the weight acts vertically.

Hence for equilibrium it is necessary that the point of suspension should be vertically above the C.G. In the figures we have two different cases of equilibrium, when the C.G. is *below* the point of suspension the equilibrium is stable, for when slightly displaced the body's weight will turn it back again. When the C.G. is *above* the point of suspension the equilibrium is unstable, for when displaced the body will not return to its former position.

If the C.G. coincides with the point of suspension the equilibrium is neutral.

132. Condition for Equilibrium when a Body is placed on a Horizontal Plane.

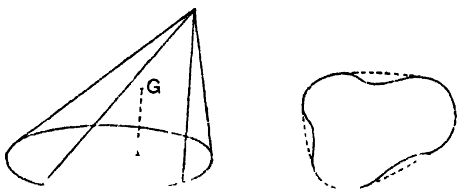

FIG. 97.

A body is placed on a smooth horizontal plane, it is required to find the condition that it should not fall over.

In order that it may rest, the resultant pressure of the (vertical) pressures at the points where it is in contact with the plane, must just counterbalance the weight of the body (which acts at its C.G.).

But the line of action of this resultant pressure passes through some point *within* the base of the body, hence, *the vertical line through the C.G. must cut the base.* If this line does not cut the base the body will fall over. By the *base* is meant the area enclosed by a string drawn tightly round the body where it meets the plane, this is sometimes greater than the area in contact with the plane, as in the second figure.

133. Equilibrium of Forces acting at a Point.

The condition of equilibrium of forces which act at a point is that the algebraic sum of the resolved parts of the forces in each of two different directions should be zero.

For the sum of the resolved parts of the forces in any direction is equal to the resolved part of their resultant in that direction. See Art. 56.

And the resolved part of a force is zero in only *one* direction (viz. the direction perpendicular to it).

Hence if the resultant force has its resolved parts in two different directions each zero it must be itself zero.

Another way of stating that the resultant should be zero has been already given, viz. that the force-polygon should be closed. Art. 47.

Notice that having once ascertained a set of forces to be in equilibrium, we know that the algebraic sum of their resolved parts in *any* direction is zero, since it equals the resolved part of the (zero) resultant in that direction.

Ex. Four forces act at a point O, viz. forces of 8, 10 and 12 lbs. making angles of 30°, 60° and 150° respectively with a line OA, and a force of 4 lbs. making an angle of 30° with OA measured in the opposite direction. Find what forces must act along and perpendicular to OA to produce equilibrium.

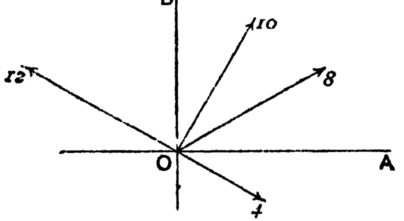

FIG. 98.

The components along OA of the forces are

$$8 \times \frac{\sqrt{3}}{2}, \; 10 \times \frac{1}{2}, \; -12 \times \frac{\sqrt{3}}{2}, \; 4 \times \frac{\sqrt{3}}{2}.$$

These are equivalent to a force along OA of 5 lbs.

The components perpendicular to OA are

$$8\times\frac{1}{2},\ 10\times\frac{\sqrt{3}}{2},\ 12\times\frac{1}{2},\ -4\times\frac{1}{2}.$$

These are equivalent to a force of $8+5\sqrt{3}$ lbs.

To produce equilibrium we must have forces -5 lbs., $-8-5\sqrt{3}$ lbs. along and perp. to OA.

134. Equilibrium of any Number of Forces in One Plane.

The conditions of equilibrium for any number of forces whose lines of action lie in one plane, are as follows:

(i) The algebraic sum of their resolved parts in some two directions must vanish.

(ii) The algebraic sum of their moments about some point in the plane must vanish.

For any forces in one plane can be replaced by either a *force* or a *couple*. Art. 105.

The given forces cannot be replaced by a *force*, because in that case (i) would not be satisfied. since the resolved part of a force is zero only in the direction perpendicular to itself.

The given forces cannot be replaced by a *couple*, because then (ii) would not be satisfied, since the moment of a couple about every point in its plane is the same and not zero.

Hence the given forces have neither a resultant force nor a resultant couple and are therefore in equilibrium, and the above conditions are *sufficient*.

Cor. 1. If one point in the body is fixed the condition of equilibrium is that the resultant of all the forces should pass through the fixed point.

For then there will be no rotation, which is the only possible motion.

Cor. 2. In applying the above conditions of equilibrium we shall notice that the work is simplified by resolving the forces in directions perpendicular to forces not required to be found, and by taking moments about some point on the line of action of a force which is not required to be found.

Observe that when forces *are* in equilibrium, *i.e.* have no resultant force or couple, conditions (i) and (ii) are satisfied for *any* direction and *any* point.

135. Second set of conditions of Equilibrium.

The following three conditions of equilibrium are an alternative to those of the last Article:

A set of forces in one plane are in equilibrium if the sum of their moments is zero about each of three points in the plane which are **not in one straight line.**

For any set of forces can be replaced by either a force or a couple.

The given forces cannot be replaced by a force, since its line of action cannot pass through all the points, which it would have to do in order that its moment about each should be zero. The given forces cannot be replaced by a couple, since the moment of a couple is zero about no point.

Hence the forces must be in equilibrium. Similarly we may prove, as conditions of equilibrium, the following:

The sum of the moments about each of two points should vanish, and the sum of the resolved parts vanish in some one direction **not perpendicular** *to the line joining the points.*

The student should notice carefully that *if* the forces *are* in equilibrium, the sum of their moments about *any* point is zero, and the sum of their resolved parts in *any* direction is zero.

136. Examples.

The use of the above conditions is best understood from examples; we add the following to illustrate their use.

Ex. 1. A ladder AB rests against the side of a house and is inclined at 60° to the ground. The pressure of the ladder against the wall being equal to a force of 60 lbs. and the friction at the same place being equal to a force of 40 lbs., find the vertical pressure and friction at the point where the ladder rests on the ground. Find also the weight of the ladder.

The forces at the foot of the ladder are the vertical pressure R and the force of friction F, notice that F acts horizontally and its direction

is opposite to that in which motion would occur if there were no friction. Let W be the weight of the ladder.

The sum of the horizontal forces must be zero, hence

$$60 - F = 0 \quad \text{or} \quad F = 60 \text{ lbs. wt.} \qquad \text{(1)}.$$

Also the sum of the vertical forces is zero, hence

$$R + 40 - W = 0 \qquad \text{(2)}.$$

The algeb. sum of the moments of the forces about B, the foot of

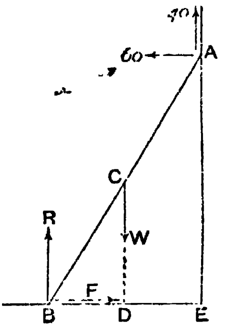

Fig. 99.

the ladder, is zero, hence if l be the length of the ladder, the moment of W about this point is

$$W \cdot BD = W \times \frac{1}{2} \times \frac{l}{2} = W\frac{l}{4};$$

this moment is *negative*.

The moments of the forces at A are

$$60 \times AE \text{ and } 40 \times BE, \text{ or } 60 \times \frac{\sqrt{3}l}{2} \text{ and } 40 \times \frac{l}{2};$$

these moments are *positive*. Hence taking the sum

$$60\frac{\sqrt{3}l}{2} + 40\frac{l}{2} - W\frac{l}{4} = 0 \qquad \text{(3)},$$

or $$W = 40\,(2 + 3\sqrt{3}).$$

Hence inserting this value in (2) we get

$$R = 40\,(1 + 3\sqrt{3}).$$

Observe that F and R have no moment about B, since they both pass through it.

Ex. 2. A rod AB can turn freely about a pivot fixed in a wall at A, and is supported at B by a horizontal string which is fastened to a point in the wall at C. Find the tension of the string and the reaction at the point.

The distance AC is 8 inches and BC 6 inches.

Let the reaction R of the pivot have for its horizontal and vertical components the forces X and Y, let T be the tension of the string and W the weight of the rod.

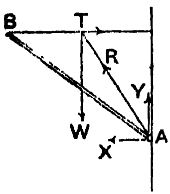

FIG. 100.

Since the sum of the horizontal forces acting on the rod is zero,

$$X - T = 0, \text{ or, } X = T,$$

since the sum of the vertical forces acting on the rod is zero,

$$Y - W = 0, \therefore Y = W.$$

Hence $R = \sqrt{X^2 + Y^2} = \sqrt{T^2 + W^2}$.........(1).

Again, the sum of the moments of the forces about A is zero, therefore

$$T \times 8 - W \times 3 = 0,$$

(since the perp. distance of W from A is $= \frac{1}{2} BC$)

or, $T = \frac{3}{8} W$..................................(2).

Substituting in (1) from (2) we get

$$R = \sqrt{\frac{9}{64} W^2 + w^2} = \frac{\sqrt{73}}{8} W = 1 \cdot 07\, W \text{ nearly}.$$

Notice that the direction of R must pass through the intersection of the forces T and W

Ex. 3. A uniform rod AB is pivoted at C and rests on a peg at D, C and D being equidistant from its middle point.

Find the least weight placed at one end which will just lift it off the peg D.

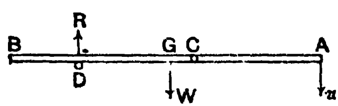

FIG. 101.

Let W be the weight of the rod, w the weight placed at the end A, R the reaction at D, and the distance of A and C from G the middle point 12 and 2 inches respectively.

Take moments about C, we have

$$10w - 2W + 4R = 0.$$

Now when the weight w is just sufficient to take the rod from contact with D we have $R=0$;

$$\therefore\ 10w-2W=0, \quad \text{or} \quad w=\tfrac{1}{5}W.$$

Ex. 4. Two equal weights W connected by a string hang over two smooth pegs A and B in the same horizontal line; neglecting the weight of the string find the pressure on each peg.

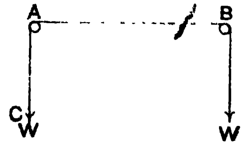

FIG. 102.

The tensions in the portions of string CA and AB are equal, hence the pressure on the peg A is the resultant of two equal forces each equal to W acting at right angles; this resultant equals $\sqrt{2}\,W$ which is therefore the pressure on the peg A and similarly is also the pressure on the peg B.

Ex. 5. A solid cube of wood rests on a table, one of its lower edges AB is fixed to the table so that the cube can turn round it. The cube is pulled by a horizontal string which is fastened to the middle point of the edge CD, the string being perpendicular to CD. Find the least tension in the string which will just make the cube begin to turn about AB. The length of an edge of the cube is 10 inches.

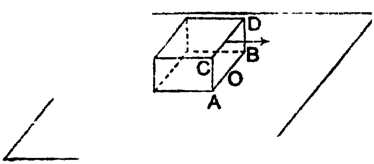

FIG. 103.

When the cube is just being moved it will only touch the table along AB. The forces on the cube are, its weight, the tension of the string and the reaction of the edge AB, which will act at the middle point O of AB and in the plane passing through the string and the centre of the cube.

Take moments about O, then if T is the tension of the string

$$T\times 10 - W\times 5=0, \quad \text{or} \quad T=\tfrac{1}{2}W.$$

Ex. 6. Two circular cylinders of the same length are placed at the bottom of a long box, in contact, and upon them a third cylinder is

placed. Find the pressure between the upper and lower cylinders, and the pressures between the sides of the box and the lower cylinders, there being no mutual pressure between the lower cylinders.

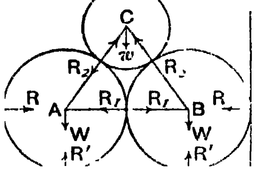

FIG. 104.

The figure represents the vertical section through the middle of the cylinders. The pressures along the lines of contact have resultants in this plane, and are represented in the figure.

The centres of the sections are A, B and C; the mutual pressure of the upper and lower cylinders is R_2, of the lower cylinders with the sides and bottom of the box are R and R' respectively.

It is given that the lower cylinders have no mutual pressure, or that $R_1 = 0$. Let the angle CAB be a; W, W, and w the weights of the cylinders.

For the equilibrium of the upper cylinder,

$$2R_2 \sin a = w, \quad \therefore \quad R_2 = \frac{w}{2 \sin a}.$$

For either of the lower cylinders,

$$R - R_2 \cos a = 0, \quad \therefore \quad R = \tfrac{1}{2} w \cot a,$$

$$R_2 \sin a + W - R' = 0, \quad \therefore \quad R' = W + \frac{w}{2}.$$

Ex. 7. A string passes over two smooth pegs at A and B and sustains equal weights W at its ends, a weight w is attached to a small ring which slides on the part of the string between A and B; when the system has taken up a position of equilibrium find the inclination of the bent parts of the string to the vertical.

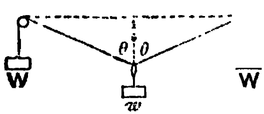

FIG. 105.

If θ is this inclination, since the tension throughout and equal to T, say,

$$2T \cos \theta = w,$$

$$T = W,$$

hence

$$\cos \theta = \frac{w}{2W}.$$

Ex. 8. A rod AB of length $2a$ and weight w is connected with a joint A at one end, a string of length l is fastened to the joint and to a point on the surface of a smooth sphere of radius r and weight W.

The rod and sphere lean against each other; in the position of equilibrium find the inclination of the rod to the vertical and the tension of the string.

Let R be the mutual pressure of the sphere and rod, then since the sphere is smooth the line of action of R passes through the centre O of

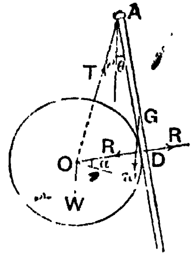

FIG. 106.

the sphere, at which W also acts, hence by Art. 129 the direction of the string passes through O.

Let T be the tension of the string, ϕ the angle it makes with the vertical, θ the angle the rod makes with the vertical.

The rod is in equilibrium under the action of R, w and a force at A.

Now the sum of the moments of these forces about A must vanish, hence

$$R \times AD = w \times \frac{AB}{2} \sin\theta, \text{ or}$$

$$R = w\,\frac{a \sin\theta}{\sqrt{(l+r)^2 - r^2}} = \frac{wa \sin\theta}{\sqrt{l^2 + 2lr}} \qquad \text{.....................(1)}$$

Again, consider the equilibrium of the sphere, the algebraic sum of the resolved parts of the forces parallel to AD must vanish, hence if α is the angle OAD, or $\theta+\phi$,

$$T\cos\alpha - W\cos\theta = 0 \qquad \text{.....................(2).}$$

Similarly resolving perpendicularly to OA,

$$R\cos\alpha - W\sin\phi = 0 \qquad \text{.....................(3).}$$

We have one more equation which expresses the fact that $\theta+\phi$, or α, is known, for

$$\tan\alpha = \frac{r}{\sqrt{l^2+2lr}} \qquad \text{.....................(4).}$$

The last equation is called a *geometrical* equation.

The equations (1), (2), (3) and (4) enable us to find the four quantities T, R, θ, and ϕ. Thus from (1) and (3),

$$w\frac{a\sin\theta}{\sqrt{l^2+2lr}}=W\frac{\sin\phi}{\cos a}=W\frac{\sin(a-\theta)}{\cos a},$$

or, from (4),

$$w\frac{a}{r}\tan a\sin\theta=W\frac{\sin(a-\theta)}{\cos a}, \text{ hence}$$

$$\tan\theta=\frac{W\tan a}{w\frac{a}{r}\tan a+W}.$$

Thus θ is found, and hence T, which is equal to $W\frac{\cos\theta}{\cos a}$.

EXAMPLES. XXXI.

1. A square is capable of motion in a vertical plane round an angular point, and a weight half that of the square is suspended at an adjacent angular point; find the position of equilibrium.

2. A ball of radius one foot and weight 8 lbs. is fastened by a string attached to a point on its surface and 8 inches long to a smooth vertical wall; find the pressure on the wall and the tension of the string.

3. Two equal heavy spheres of 1 inch radius are in equilibrium within a smooth spherical cup of 3 inches radius. Show that the pressure between the cup and one of the spheres is double the pressure between the two spheres.

4. If forces A, B, C, D acting along the sides a, b, c, and d of a quadrilateral are in equilibrium, prove that

$$\frac{AC}{ac}=\frac{BD}{bd}.$$

5. A triangle ABC whose weight is 33 lbs. lies partly on a table, the vertex C projecting beyond the edge a distance of 10 inches; if the distance of the C.G. from the edge of the table is 2 inches, find the least weight which placed at C would overturn it.

6. A uniform equilateral triangular lamina ABC of 3 lbs. weight can turn in a vertical plane about a hinge at B; it is supported with the side AB horizontal by a smooth prop at the middle point of BC. Find the pressure on the prop and the reaction at the hinge.

7. A pair of compasses each of whose legs is a uniform bar of weight W is supported, hinge downwards, by 2 smooth pegs at the middle point of the legs in the same horizontal line, the legs being kept apart at the angle A by a weightless rod joining their extremities; find the thrust on the rod.

8. A uniform sphere rests on a smooth inclined plane and is supported by a horizontal string. To what point of the sphere must the string be attached?

9. Two equal weights W are attached to the extremities of a thin string which passes over three tacks in a wall arranged in the form of an isosceles triangle with the base horizontal, the vertical angle at the upper tack being 120°, find the pressure on each tack.

10. A straight rod two feet long is moveable about a hinge at one end and is kept in a horizontal position by a thin vertical string attached to the rod at a distance of 8 inches from the hinge and fastened to a fixed point above the rod; if the string can just support a weight of 9 ozs. without breaking, find the greatest weight that may be suspended from the other end of the rod, the rod weighing 2 ozs.

11. A thin ring of radius R and weight W is placed round a vertical cylinder of radius r and is prevented from falling by a nail projecting horizontally from the cylinder. Find the pressure between the cylinder and the ring.

12. A piece of wire of weight 4 ozs. is bent to form the sides AC, CB of an equilateral triangle, a weight of 1 oz. is attached to B, and the wire is suspended from A, show that in the position of equilibrium BC is horizontal.

13. A picture is to hang at a given inclination α to a wall by a string attached to the middle of the top side and to the wall. Find the inclination of the string to the wall, having given that the reaction makes an angle of 45° with the wall.

14. A rod of length a hangs against a smooth vertical wall suspended by a string of length l tied to one end of the rod and to a point in the wall; prove that the rod may rest inclined to the vertical at an angle θ where $\cos^2\theta = \dfrac{l^2 - a^2}{3a^2}$.

15. A rod of weight W rests against a smooth inclined plane AB and a smooth vertical plane BD (B being higher than A). A string is attached to the lower end of the rod, and passing over a pulley at B supports a weight P. If θ be the inclination of the rod (in equilibrium) to the horizon and α that of the plane, $\cot\theta = 2\sec\alpha\left(\dfrac{P}{W} - \sin\alpha\right)$.

16. A heavy rectangular board $ABCD$ of uniform thickness and of weight W is supported at the middle point of AB, about which it can turn freely in a vertical plane. Weights P, Q ($P > Q$) are attached by strings to the corners B, D and hang freely. Prove that in the position of equilibrium the inclination of AB to the horizon will be

$$\tan^{-1}\left(\frac{P-Q}{2Q+W}\frac{a}{b}\right),$$

where a and b are the lengths of the sides AB, AD of the rectangle.

17. A hollow vertical cylinder, radius $2a$, height $3a$, rests upon a horizontal table; a rod is placed within it with its lower end at the circumference of the base, the rod rests upon the opposite point of the upper rim and projects over. How long must the rod be in order that it may cause the cylinder to topple over, the rod and cylinder being of equal weight?

18. A heavy uniform rod of length $2a$ rests partly within and partly without a smooth fixed hemispherical bowl of radius r; if θ be the inclination to the horizon of the rod

$$2r \cos 2\theta = a \cos \theta.$$

19. A heavy triangular lamina rests inside a smooth hemispherical bowl; prove that the pressures at the three angular points are equal.

20. A uniform beam rests with one smooth end against the junction of the ground and a vertical wall; it is supported by a string fastened to the other end of the beam and to a staple in the vertical wall. Find the tension of the string, and show that it will be equal to half the weight of the beam if the length of the string is equal to the height of the staple above the ground.

21. A cylindrical tube of mass M stands upright on the ground. Two equal smooth spheres are placed within it, one resting on the ground and the other supported by the cylinder and the other sphere. If the mass of either sphere be m, its radius a, and the radius of the cylinder $\frac{3}{2}a$, show that the cylinder is on the point of toppling over provided that $2m = 3M$.

22. Three equal strings of no sensible weight are knotted together to form the equilateral triangle ABC, and a weight W is suspended from A. If the triangle and weight be supported with BC horizontal by means of two strings, each at the angle of 135° to BC, prove that the tension in BC is $\frac{W}{6}(3 - \sqrt{3})$.

23. Two equal weights of 112 lbs. are joined by a string passing over two pulleys A and B in the same horizontal line. A weight of 1 lb. is attached to the middle point of the string between A and B; find the position of equilibrium.

24. Three uniform heavy rods AB, BC and CA of lengths 5, 4 and 3 feet respectively are hinged together at their extremities to form a triangle. Prove that the whole will balance with AB horizontal about a point distant $1\frac{1}{5}$ of an inch from the middle point of AB towards A.

25. If W be the total weight of the 3 rods in the last question prove that the vertical components of the action at the hinges A and B when the rod is balanced are $\frac{187}{360}W$ and $\frac{183}{360}W$.

26. Show that if the pins of the two hinges of a door are not in the same straight line the door cannot open, and if the straight line of the pins is not vertical the door will tend either always to open or always to close.

27. If a system of forces be represented in magnitude and position by all but one of the sides of a closed polygon taken in order, their resultant will be parallel in position and proportional in magnitude to the remaining side, and that its line of action will be at a distance from that side inversely proportional to its length.

28. A uniform heavy rod AB, 4 feet long, whose weight is 10 lbs., passes under a smooth peg 6 inches from the end A and over another peg 12 in. from the same end. A weight of 100 lbs. is suspended from the end B; find the actions on the pegs.

29. A rectangular lamina $ABCD$, in which the angle BAC is 60°, can turn freely in a vertical plane about its corner A which is fixed, and a string attached to B passes over a small smooth pulley fixed vertically above A at a distance equal to AB and sustains a weight equal to the weight of the board. Show that in equilibrium with D lower than B the inclination of AB to the vertical is either 60° or 20°.

CHAPTER IX.

THE SIMPLE MACHINES.

137. THE instrument by which effort is applied to lift a weight or overcome any kind of resistance is called a *machine*. Part of the effort is spent in overcoming the resistance of the machine itself due to friction, imperfect flexibility of ropes, &c. Resistances of this kind are called *wasteful* as distinguished from that which it is the object of the machine to overcome, this latter is called *useful* resistance.

138. Efficiency.

The ratio of the *useful* work done by a machine to the whole work done is called its efficiency. If there were no wasteful resistance the efficiency would be unity. In such a case the machine is said to be perfect, and this we shall assume it to be.

139. Mechanical advantage.

In what follows it is supposed that an applied force P, by means of a machine, supports a weight Q, then any force greater than P will move Q. The ratio $\frac{Q}{P}$ is called the *mechanical advantage* of the machine and is usually greater than unity.

140. Simple Machines.

The simpler machines are the following:

(1) The Lever,
(2) The Wheel and Axle,
(3) The Pulley,
(4) The Inclined Plane,
(5) The Screw.

The Lever.

The lever consists of a rigid bar (straight or curved) which can turn freely about a fixed axis called the *fulcrum.*

By its means an applied force P balances a force Q, a pressure R being produced on the fulcrum.

There are three classes of levers; the *first* is when the fulcrum is between P and Q,

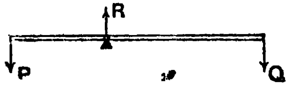

FIG. 107.

the *second,* when Q acts between P and R,

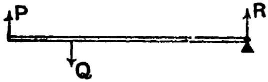

FIG. 108.

the *third,* when P acts between Q and R.

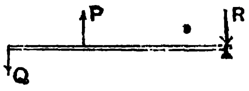

FIG. 109.

Examples of the first kind are, a pair of scales, a crowbar, a poker resting on the bar of a grate; *double* levers are a pair of scissors.

Examples of the second kind are, an oar when the end in contact with the water is at rest, this end is the fulcrum, the force Q acts at the rowlock; a wheelbarrow, the fulcrum being the point of contact with the ground.

An example of the third kind is the limb of an animal; the socket is the fulcrum, the force P is the action of a muscle attached to the bone near the socket, the force Q is the weight of the limb; a pair of tongs is an example of a double lever of this class.

141. Conditions of Equilibrium.

For equilibrium R must be the reversed resultant of the parallel forces P and Q, hence in all cases

$$P \cdot AC = Q \cdot BC. \qquad \text{Art. 89.}$$

Also, in the first case $R = P + Q$,

in the second case $R = Q - P$,

in the third case $R = P - Q$.

If the lever is bent, draw from C lines CM and CN perpendicular to the directions of P and Q, then taking moments about C,

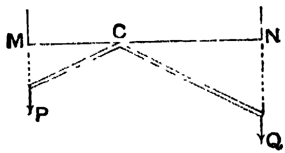

Fig. 110.

we have $$P \cdot CM = Q \cdot CN.$$

142. In the foregoing we have neglected the weight of the lever, if it is of weight w and its C.G. at O,

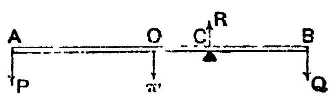

Fig. 111.

by taking moments about C as before, we have

$$P \cdot AC + w \cdot OC = Q \cdot BC.$$

143. The distances AC and BC are called the "arms" of the lever;

and since mechanical advantage $= \dfrac{Q}{P} = \dfrac{AC}{BC}$, Art. 139,

we see that the mechanical advantage is $= \dfrac{P\text{'s arm}}{Q\text{'s arm}}$.

Class I. Mechanical advantage is gained by making P's arm longer than Q's arm.

Class II. Mechanical advantage is always gained.

Class III. Mechanical advantage is always lost.

If the lever gives mechanical advantage we see that the distance moved through by P is greater than that moved through by W.

Ex. 1. Two weights balance attached to the ends of a lever of the first class, one weight is 10 lbs. and its arm is 6 inches, the other arm is 15 inches, find the other weight.

Let Q be the other weight, here $P=10$, hence

$$10\times6=Q\times15, \quad \text{or} \quad Q=4 \text{ lbs.}$$

Ex. 2. The mechanical advantage of a lever is 4 and the pressure on the fulcrum is 8 lbs., find the balancing forces, the lever being of the second class.

We have given that $\quad \frac{Q}{P}=4, \quad Q-P=8.$

Hence $\quad P=\frac{8}{3}$ lbs., $\quad W=\frac{32}{3}$ lbs.

Ex. 3. A man who wishes to raise a rock leans with his whole weight on one end of a crowbar five feet long which is propped at the distance of four inches from the end in contact with the rock. The man's weight being 160 lbs., what force does he exert on the rock, and what is the pressure on the prop?

The force he exerts is equal to that of a weight x, where x is given by the equation

$$4x=160\times56, \quad \text{or} \quad x=2240 \text{ lbs.}$$

The pressure on the prop is

$$=W+P=2400 \text{ lbs.}$$

Ex. 4. Two forces whose measures are 6 and 8 act at the end of a rod 16 feet long and are inclined to the rod at angles of 30° and 45° respectively; find the position of the fulcrum when there is equilibrium.

Let AB be the rod and x the distance of the fulcrum from A.

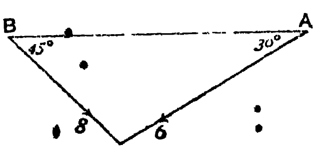

FIG. 112.

The perpendicular distance of the force 6 from the fulcrum is $\frac{x}{2}$.

The distance of the fulcrum from B is $16-x$.

The distance of the force 8 from the fulcrum is $\frac{16-x}{\sqrt{2}}$.

$$\therefore\quad 6\times\frac{x}{2}=8\times\frac{16-x}{\sqrt{2}},$$

hence $$\frac{3x}{\sqrt{2}}=64-4x,\quad\text{or}\quad x=10{\cdot}4 \text{ feet nearly.}$$

EXAMPLES. XXXII.

1. The arms of a straight lever are 5 and 7 inches long respectively, a weight of 2 lbs. is attached to the shorter arm, find the weight to be attached to the longer arm for equilibrium.

2. A weightless rod 7 feet long has weights of 4 lbs. and 10 lbs. hung at its extremities. Find the position of the fulcrum when there is equilibrium.

3. A uniform rod which is 16 feet long and which weighs 17 lbs. can turn freely about a point in itself, the rod is in equilibrium when a weight of 7 lbs. is hung at one end. Find the position of the fulcrum.

4. The two arms of a straight lever are 18 inches and 50 inches long respectively, and its weight is 10 lbs. If a weight of 58 lbs. be applied at the end of the longer arm, what weight must be applied at the end of the other that there may be equilibrium?

5. A straight lever whose length is 5 ft. and weight 10 lbs. has its fulcrum at one end. Weights of 3 lbs. and 6 lbs. are fastened to it at distances 1 ft. and 3 ft. from the fulcrum, and it is kept horizontal by a force at the other end. Find the pressure on the fulcrum.

6. If the pressure on the fulcrum be equal to 10 times the difference of the forces, find the ratio of the arms.

7. A uniform wire is bent so as to form two straight lines inclined at an angle of 120°, one of which is twice as long as the other. The wire is suspended from its angular point. Find the position of equilibrium.

8. A uniform heavy rod has a weight of 5 lbs. hung from one end and balances on a fulcrum 5 feet from that end. This weight is replaced by a weight of 10 lbs. and then the rod balances when the fulcrum is 4 feet from that end. Find the length and weight of the rod.

9. A man seated in a boat pulls at the handle of each of a pair of sculls with a force of 25 lbs. weight. If the distance of the rowlock from the end of the blade of each scull be 4 times that of the rowlock from the hand, find the resultant force on the boat.

10. The arms of a bent lever are at right angles to one another and are in the ratio of 5 to 1. The longer arm is inclined to the horizon at an angle of 45°, and carries at its end a weight of 10 lbs.; the end of the shorter arm presses against a horizontal plane, find the pressure on the plane.

144. Balances.

One of the uses of the lever is to determine weights, for this purpose it is used in the forms of the Common Balance, the Roman steelyard, and the Danish steelyard.

145. The Common Balance.

This consists of a beam somewhat in the shape of a lozenge which turns freely about a fulcrum C, consisting of a wedge-shaped knife-edge attached to the beam and resting on a fixed support. The centre of gravity G of all the parts of the balance lies vertically below C when the

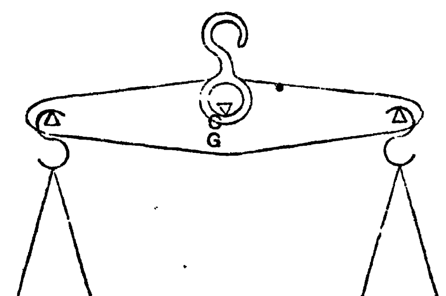

FIG. 113.

beam is horizontal. To the ends of the beam there are attached knife-edges upon which scale-pans are supported.

146. If the weights placed in the scale-pans are not equal the beam will not be horizontal, and will take up a

position of equilibrium in which the moments of the weights about C will be counterbalanced by the moment of the balance itself about C.

147. A pointer or rod CD (not shown in the figure) is often attached to the centre of the beam. In the position of equilibrium this is vertical.

148. Requisites of a good Balance.

1. The balance must be *true*, that is when loaded with equal weights the beam should be horizontal. This condition is secured if,

   (i) the arms are equal in length,

   (ii) the scale-pans are equal in weight,

   (iii) the c.g. of the beam be vertically below C.

2. The balance should be *sensitive*, that is for any small difference in the weights the deviation of the beam from its horizontal position should be large enough to be easily observed.

   This will be secured by

   (i) increasing the length of the beam AB,

   (ii) decreasing the length of the rod CD,

   (iii) diminishing the weight of the beam.

See Ex. XXXIII, 8.

3. The balance should be stable. That is when disturbed it should quickly return to its horizontal position.

149. False Balances.

The effect of weighing with an untrue balance will be now considered.

1. The arms of a balance are of unequal length a and b, a body of weight w lbs. weighs w_1 lbs. when placed in one scale and w_2 lbs. when placed in the other. Here we have

$$wa = w_1 b,$$

$$w_2 a = wb,$$

$$\therefore \quad w^2 = w_1 w_2,$$

$$w = \sqrt{w_1 w_2} \quad \text{.....................(1).}$$

The true weight is the geometric mean of the apparent weights.

2. The arms are of equal length but the C.G. of the balance is at a horizontal distance c from C, the middle point of the beam.

Let w' be the weight of the balance, w the true weight of a body and w_1, w_2 the weights which have to be used to balance it when it is placed in the two scale-pans in succession.

Here if a is the length of either arm, by taking moments about C,

$$wa + w'c = w_1 a,$$

$$w_2 a + w'c = wa.$$

Hence $$(w - w_2)\, a = (w_1 - w)\, a,$$

or $$2w = w_1 + w_2 \quad \text{..................(2)}.$$

The true weight is the arithmetic mean of the apparent weights.

3. If the arms are of unequal length and *also* the C.G. of the balance not vertically below C, we have by taking moments as before

$$w \,.\, a = w'c + w_1 b,$$

$$w_2 a = w'c + wb\,;$$

$$\therefore \quad w\,(a + b) = w_1 b + w_2 a \quad \text{..............(3)}.$$

EXAMPLES. XXXIII.

1. The arms of a balance are respectively $8\frac{3}{4}$ and 9 inches long, the goods to be weighed being suspended from the longer arm. Find the real weight of goods which apparently weigh 27 lbs.

2. A body the weight of which is 28 ounces, when placed in one scale of a balance with unequal arms, appears to weigh 14 ounces; find its weight when placed in the other scale.

3. The apparent weights of a body are 4 and 16 lbs. respectively, when weighed from the two arms of a balance. Find the ratio of the lengths of the arms and the true weight of the body.

4. If the beam of a false balance is uniform and heavy, show that the arms are proportional to the differences between the true and apparent weights.

5. In a given balance it is found that 51·075 grammes in one scale balance 51·362 in the other, and 25·592 balance 25·879; show that the arms are equal, but that the scale-pans differ by ·287 grammes.

6. The beam of a balance is 6 feet long and it appears correct when empty, a certain body placed in one scale weighs 120 lbs., when placed in the other, 121 lbs. Show that the fulcrum must be distant about $\frac{1}{13}$ of an inch from the centre of the beam.

7. A dealer has correct weights but one arm of the balance is shorter than the other. If he sells two quantities of a certain drug, each apparently weighing $9\frac{1}{2}$ lbs. at 40*s.* per lb. weighing one in one scale and one in the other scale, will he gain or lose?

8. If in Art. 145, the length of the beam is $=2a$, $CD=h$, $CG=k$, and θ the angle which the beam makes with the horizon when the two weights P and Q are placed in the pans, show that, if W is the weight of the beam,

$$\tan\theta = \frac{(P \sim Q)\,a}{(P+Q)\,h + Wk}.$$

9. If the arms of a false balance be without weight, and one arm longer than the other by $\frac{1}{9}$ of the shorter arm, and if in using it the substance to be weighed is put as often in the one scale as in the other, show that the seller loses $\frac{5}{9}$ per cent. in his transactions.

150. The Roman Steelyard.

This steelyard is a lever of the first class and consists of a bar moving about a fulcrum C and having a given weight w sliding on the longer arm.

To the end of the shorter arm the body whose weight is required is attached and the weight w is moved along its arm until there is equilibrium.

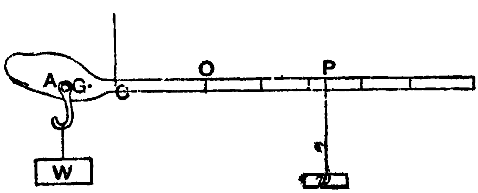

Fig. 114.

The longer arm is graduated so that the point at which w is placed indicates the weight of the body.

To graduate the Roman steelyard.

The point from which the graduations start must be where w is placed to balance the weight of the steelyard itself when *no* weight is attached. Let this point be O. Then W' being the weight of the steelyard

$$w \,.\, CO = W' \,.\, CG \text{..................(i).}$$

Again, if a weight W is attached and there is equilibrium when w is moved to P, taking moments about C, we have

$$w \,.\, CP = W' \,.\, CG + W \,.\, AC \text{..............(ii).}$$

By subtracting (i) from (ii) we get

$$w \,.\, OP = W \,.\, AC, \text{ or } \frac{W}{w} = \frac{OP}{AC} \text{ ....(iii).}$$

Hence if OB be divided up into portions each equal to AC, starting from O, the graduation at which w rests gives the number of times W contains w.

If w is one lb. the graduations indicate pounds. The graduations are all at equal distances AC and if we wish to be correct to ounces each graduation must be divided into 16 equal parts.

If the centre of gravity G of the bar be in the longer arm the point O will be found to be in the shorter arm.

Ex. 1. If the fulcrum be four inches from the point to which the weight is attached, and the centre of gravity five inches from that end, and the weight of the bar equal to the moveable weight, find the position of zero graduation.

We have given that $W' = w$, also G is one inch from C and in the longer arm, hence O is also one inch from C and in the shorter arm.

Ex. 2. The moveable weight is originally one pound and a weight of three pounds is substituted, the graduations remaining the same, how is a buyer affected by the change, the C.G. being in the shorter arm?

If the moveable weight were at P the buyer is charged for the weight $3\frac{OP}{CA}$ lbs. If the graduations had been constructed for a moveable weight of 3 lbs., then instead of OP we should have had $O'P$ where O' is such that $W' \,.\, CG = 3CO'$, and the buyer *should* be charged for $3\frac{O'P}{CA}$ lbs. Hence CO' is less than CO and therefore $O'P$ greater than OP. The buyer therefore gets more for his money than he should do.

Ex. 3. The weight of a steelyard is 10 lbs., the body to be weighed is suspended from a point 4 inches from the fulcrum and the C.G. of the steelyard is 3 inches on the other side of the fulcrum. Where should the graduation corresponding to one cwt. be situated, the moveable weight being 12 lbs.?

The zero of graduation is got in this case from the equation

$$10 \times 3 = 12 \times CO \text{ by (i).}$$

Hence O is $2\frac{1}{2}$ inches from the fulcrum.

Also $$112 \times 4 = 12 \times OP, \quad \text{or} \quad OP = \tfrac{112}{3},$$

from which it follows that the point required is distant from the fulcrum

$$\tfrac{112}{3} - \tfrac{5}{2} \quad \text{or} \quad 34\tfrac{5}{6} \text{ inches.}$$

EXAMPLES. XXXIV.

1. A steelyard 4 feet long has its C.G. 11 inches and its fulcrum 8 inches from A. If the weight of the machine be 4 lbs. and the moveable weight 3 lbs., find how many inches from A is the graduation denoting 15 lbs. weight.

2. In a Roman steelyard the sliding weight is 10 lbs., the graduations for a difference of one stone are $3\frac{1}{2}$ inches apart, find how far from the fulcrum is the point at which the bodies to be weighed are attached.

3. If the moveable weight be equal to the weight of the beam, and if the zero of the graduations bisect the distance between the fulcrum and the point of suspension of the body to be weighed, then the first graduation will coincide with the C.G. of the beam.

4. If a steelyard by use lose $\frac{1}{10}$ of its weight, its C.G. remaining unaltered, show how to correct the graduations of the steelyard.

5. A steelyard is 12 inches long and with the scale-pan weighs 1 lb., the C.G. of the two being 2 inches from the end to which the scale-pan is attached. Find the position of the fulcrum when the moveable weight is 1 lb., and the greatest weight that can be ascertained by the steelyard is 12 lbs.

6. The beam is 32 inches long, the body to be weighed being attached at one end A; the fulcrum is distant 5 inches and the C.G. of the beam $7\frac{1}{4}$ inches from A. The weight of the beam being 1 lb. and that of the moveable weight $3\frac{1}{4}$ lbs., find the heaviest weight that can be weighed by this instrument.

7. A steelyard is damaged, either by rust or by losing a portion of its longer arm, causing it to weigh inaccurately, show that it may be repaired by the attachment of a suitable weight, so that we can use the original graduations.

151. The Danish Steelyard.

This consists of a bar having at one end a heavy lump of metal; to the other end the body to be weighed is attached. We then observe about what point in its length the bar balances.

Thus the fulcrum is moveable.

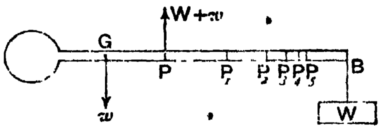

FIG. 115.

To graduate the Danish Steelyard.

Let w be the weight of the bar and G its centre of gravity.

Then if a weight W be attached, and the system balances about P, the upward force at P is $W+w$, and taking moments about B we get

$$w \, . \, BG = (W + w) \, BP,$$

or

$$BP = \frac{w}{W+w} BG.$$

Now first let $W = w$, then $BP_1 = \frac{1}{2} BG$, mark this position of the fulcrum P_1; next let $W = 2w$, then

$$BP_2 = \frac{w}{2w + w} BG = \frac{1}{3} BG,$$

mark this position P_2: in the same way taking W to be $3w$, $4w$, &c. in succession we get the points P_3, P_4 &c., where

$$BP_3 = \tfrac{1}{4} BG, \quad BP_4 = \tfrac{1}{5} BG, \text{ \&c.}$$

Thus the bar is graduated, and if for instance w is one lb., the bodies which cause the fulcrum to take the positions P_1, P_2, P_3 &c. are of the respective weights one lb., two lbs., three lbs. &c.

Observe that the distances BP_1, BP_2 &c. are in harmonical progression.

By taking W equal successively to $\frac{1}{2}w$, $\frac{1}{3}w$, $\frac{1}{4}w$, ... we get the corresponding distances of the fulcrum from B as $\frac{2}{3}BG$, $\frac{3}{4}BG$, $\frac{4}{5}BG$, ... These points should be marked $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ... Notice that their distances from G are in harmonical progression.

Ex. 1. A Danish steelyard weighs 10 lbs. and the distance of its C.G. from the scale-pan is 4 feet, find the distances of the successive points of graduation from the C.G.

By this Article $\qquad BP = \dfrac{10}{W+10} \times 4.$

When $W=1$, $BP_1 = \frac{40}{11} = 3\frac{7}{11}$ feet, or $GP_1 = \frac{4}{11}$ feet,

$W=2$, $BP_2 = \frac{40}{12} = 3\frac{1}{3}$ feet, or $GP_2 = \frac{2}{3}$ feet,

$W=3$, $BP_3 = \frac{40}{13} = 3\frac{1}{13}$ feet, or $GP_3 = \frac{12}{13}$ feet, and so on.

Ex. 2. A Danish steelyard has at one end a ball of metal 3 inches in diameter and weighing 8 lbs. The bar weighs 2 lbs. and is 12 inches long. It is graduated from end to end, the spaces being $\frac{1}{2}$ an inch apart. What are the greatest and least weights that can be measured by it?

Let x be the distance of the C.G. of the bar and ball from the centre of the ball, then

$$8x = 2\left(7\tfrac{1}{2} - x\right), \quad \text{or} \quad x = 1\tfrac{1}{2} \text{ inches.}$$

Thus the C.G. is at the end of the bar. Also by this Article

$$BP = \frac{10}{10+W} \times 12.$$

When the weight is greatest BP is least or $\frac{1}{2}$ an inch,

$$\therefore \ \tfrac{1}{2}(10+W) = 120, \quad \text{or} \quad W = 230 \text{ lbs.}$$

When the weight is least BP is greatest or $11\frac{1}{2}$ inches,

$$\therefore \ \tfrac{23}{2}(10+W) = 120, \quad \text{or} \quad W = \tfrac{10}{23} \text{ lbs.}$$

EXAMPLES. XXXV.

1. In a Danish steelyard the distance between the zero graduation and the end of the instrument is divided into 30 equal parts, and the greatest weight that can be weighed is 5 lbs. 7 ozs.; find the weight of the instrument.

2. Find the length of the scale of the steelyard whose weight is 1 lb. and in which the distance between the graduations denoting 3 and 4 lbs. is one inch.

3. In a steelyard the fulcrum rests halfway between the second and third graduations, find the ratio of the weight in the scale-pan to the weight of the instrument.

4. **The length of the scale of the steelyard is 24 inches and its weight is 7 lbs.; find the distance between the positions of the fulcrum in weighing 14 lbs. and 21 lbs. respectively.**

5. From a steelyard in which A is the point of the beam from which the scale-pan is suspended, and G is the C.G. of the steelyard and scale-pan a small particle of weight w at the middle point of AG has been broken off; show that the apparent weight of a body determined by the steelyard will be too great by $\frac{1}{2}(n-1)w$, where n is the ratio of the apparent weight of the body to that of the steelyard and scale-pan.

6. Supposing the steelyard to become coated with rust, what would you do in order to use the steelyard without altering the graduations?

7. If the weight of a Danish steelyard is 5 lbs. and the fulcrum is at a distance of 3 inches from the end for a weight of 10 lbs., show that in order to balance a weight of 15 lbs. the fulcrum must be moved $\frac{3}{4}$ of an inch.

8. If a weight W balances with the fulcrum at a distance a from the C.G. of the steelyard, the distance the fulcrum must be moved in order to restore the balance when a weight w is added is equal to

$$\frac{w^2}{W(W+2w)}a.$$

152. The Wheel and Axle.

This machine consists of two cylinders of different radii having the same axis and rigidly connected. At their ends are pivots which turn freely in fixed supports.

The weight is raised by means of a rope coiled round the smaller cylinder, the force is applied by a rope coiled in the opposite direction round the larger cylinder.

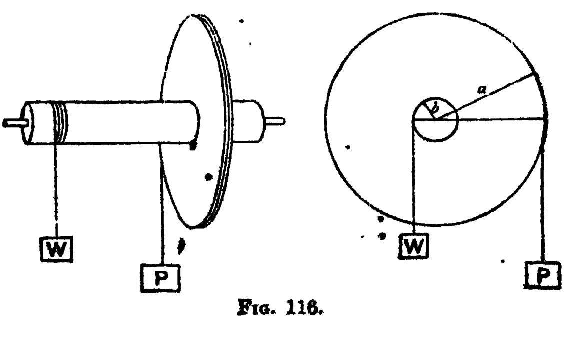

FIG. 116.

Conditions of equilibrium.

Neglecting the thickness of the ropes, take moments about the axis which gives, since the pressure on the axis has no moment,

$$P \cdot a = W \cdot b \text{.......................} (1);$$

where a and b are the radii of the wheel and axle respectively, and P is the applied force.

153. Mechanical Advantage.

The mechanical advantage, or $\frac{W}{P}$, here is

$$\frac{a}{b} = \frac{\text{radius of wheel}}{\text{radius of axle}}.$$

By making $\frac{a}{b}$ very great we can theoretically increase the mechanical advantage to any extent. Practically there are limits to the increase, if a be large the machine is unwieldy, if b be small the axle will be too weak.

When a large mechanical advantage is needed, a modified form called the Differential Wheel and Axle, which will be afterwards explained, is used.

154. The Capstan and Windlass are forms of the Wheel and Axle, the force is applied at the end of a spoke lying in a plane perpendicular to the axis. In the Capstan the axis is vertical, in the Windlass horizontal.

Ex. 1. If the radii of the wheel and axle be respectively 40 inches and 4 inches, what weight would be supported by a force equal to the weight of 30 lbs.? find also the pressures on the supports on which the axle rests.

Let x be the required weight, then

$$4x = 30 \times 40, \quad \text{or} \quad x = 300 \text{ lbs.}$$

The pressure on the supports $= P + W = 330$ lbs. weight.

Ex. 2. A wheel and axle is used to raise a bucket from a well. The diameter of the wheel is 15 inches, and while it makes 7 revolutions the bucket which weighs 30 lbs. rises $5\frac{1}{2}$ feet. Find what is the smallest force that applied to a point on the circumference can turn the wheel

Let x be the required force, then by the principle of work, Art. 69, the total work done is zero since the forces are in equilibrium, or

$$x \times 2\pi r \times 7 - 30 \times 5\tfrac{1}{2} = 0,$$

or

$$x \times \tfrac{44}{7} \times \tfrac{15}{2} \times 7 - 30 \times \tfrac{11}{2} = 0,$$

from which we find x to be $\frac{1}{2}$ a lb. weight.

Ex. 3. A weight is to be raised by means of a rope passing round a horizontal cylinder 10 inches in diameter turned by a winch with an arm $2\frac{1}{2}$ feet long. Find the greatest weight which a man could so raise without exerting a pressure of more than 50 lbs. on the handle of the winch.

Let x be the weight required. The radius of the cylinder is 5 inches. The arm of the winch, which corresponds to the radius of the wheel, is 30 inches.

$$\therefore \quad x \times 5 = 50 \times 30, \quad \text{or} \quad x = 300 \text{ lbs.}$$

EXAMPLES. XXXVI.

1. If the radius of the wheel and axle be respectively 3 feet and 4 inches, what force must be applied to raise a weight of 40 lbs.?

2. A man pushing at the end of a pole 4 feet long works a capstan whose diameter is 2 feet; with what force must he push to overcome a resistance of 600 lbs. weight?

3. If the radius of the wheel be treble that of the axle, and the force and weight are together equal to 48 lbs. weight, find the magnitude of each.

4. Four sailors each exerting a force capable of raising 116 lbs. raise an anchor by means of a capstan whose radius is 1 ft. 2 inches and whose spokes are 8 feet long (measured from the axis). Find the weight of the anchor.

5. A weight of 17 lbs. just balances a weight of 79 lbs. What will be the radius of the axle if that of the wheel is 17 inches?

6. The radii of the wheel and axle are 17 inches and 4 inches and the weight is 12 lbs., find the applied force

7. A force of 10 lbs. will raise a weight of Q lbs., a force of $\frac{Q}{4}$ lbs. will raise a weight of 10 lbs., show that the radius of the wheel is twice the radius of the axle.

8. A man exerting a force of 50 lbs. weight works a capstan. He walks 4 yards round, two feet of rope being pulled in; what is the weight raised?

9. A cage is suspended by a rope passing over an axle and a man standing in the cage draws himself and the cage up by pulling at a

rope passing over the circumference of the wheel. If the joint weight of the man and cage be 14 stone and the radius of the wheel 5 times that of the axle, the rope being pulled into the cage at the uniform rate of 2 feet a second, find the tension exerted by the man and the horse-power at which he works.

155. The Pulley.

The pulley consists of a wheel which can turn freely about an axle, the axle is attached to a framework called the block.

In a groove on the circumference of the wheel runs a string acted on by a force at each end. The pulley in fig. (i) is attached to a fixed beam above and can only turn on its axle, the lower pulley of fig. (ii) can move up and down as well as turn.

The first is called a *fixed* pulley, the second a *moveable* pulley. The tension of the rope connected with a pulley

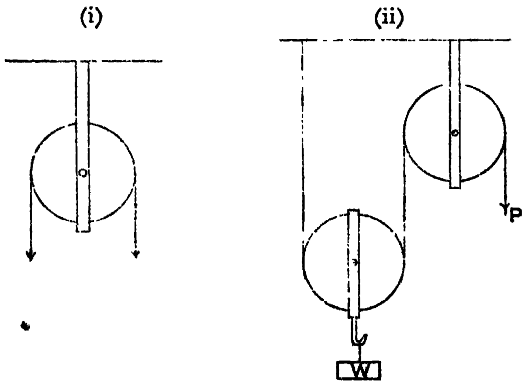

Fig. 117.

is assumed to be the same throughout its length, although this is not strictly true owing to friction. The weight of the string is negligible.

156. The *fixed pulley* has unity for its mechanical advantage, for W and P being each equal to the tension of the string are themselves equal, hence $\frac{W}{P}$ is unity.

The use of a fixed pulley is to change the direction of the applied force.

In the second figure a weight W is attached to the block, W is supported by the (equal) tensions of the string on each side, each tension is equal to P, hence $W=2P$, and the mechanical advantage is 2. The portions of the string are taken to be parallel.

157. System of Pulleys.

Pulleys are combined to form a system as represented in the figures.

The upper pulleys are fixed, the blocks of the lower ones are joined together.

The same string passes round all the pulleys, to one end of it a force P is applied, its other end is fastened to the upper block. In the case when there is one more pulley above than below, the end is fastened to the lower block.

In practice the two sets of pulleys turn on the same axes. Here the strings cannot be strictly parallel, but the error is small enough to be disregarded.

The attached weight is W, the weight of the lower block of pulleys is w, and n is the number of strings connected with the lower block.

The whole weight $W+w$ is supported by the tensions of n parallel strings, each tension is equal to P being the same throughout the string, hence

$$W+w=nP.$$

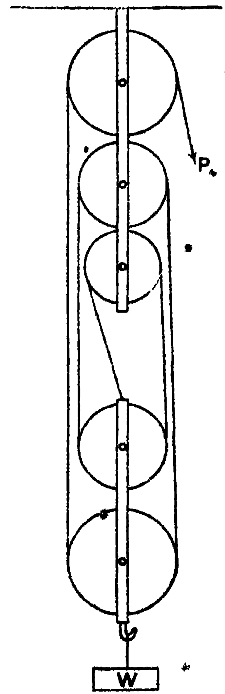

FIG. 118.

If w is negligible $W=nP$, and the mechanical advantage is n.

158. Principle of Work.

If the lower block receive an upward displacement equal to h, each portion of the string is slackened, and P descends through a space h for each portion of string, that is, through a total distance nh, if n is the entire number of strings in the lower block.

Hence by the Principle of Work, Art. 69, since the forces are in equilibrium

$$Wh - Pnh = 0, \quad \text{or} \quad W = nP.$$

Since the space traversed by the lower block is always $\frac{1}{n}$th of that traversed by the applied force, the acceleration of the lower block is $\frac{f}{n}$, where f is the acceleration of the point of application of the force P in the case when W is not equal to nP.

Ex. 1. What force will be necessary to support a weight of 32 lbs., there being nine pulleys each weighing one lb.?

Here since the number of pulleys is odd, the number of parts of the rope at the lower block is odd and the rope is attached to the lower block. There are thus 4 pulleys in the lower block.

The entire weight supported is 36 lbs.

$$\therefore \quad 9P = 36, \quad \text{or} \quad P = 4 \text{ lbs.}$$

Ex. 2. If a weight of 10 lbs. support a weight of 18 lbs., and a weight of 11 lbs. support a weight of 20 lbs., find the number of strings and the weight of the lower block.

Let n be the number of strings and w the weight of the lower block

$$18 + w = 10n,$$

$$20 + w = 11n.$$

Hence $$n = 2, \quad w = 2.$$

Ex. 3. Find the smallest weight which can be raised with mechanical advantage if there are 10 pulleys, each pulley weighing 4 lbs.

Here there are 5 pulleys on the lower block, the total weight is

$$W + 20.$$

$$\therefore \quad W + 20 = 10P.$$

For mechanical advantage greater than unity

$$W \text{ must be greater than } P,$$

$$10W \quad ,, \quad ,, \text{ greater than } W + 20,$$

$$W \text{ must be greater than } \tfrac{20}{9} \text{ lbs. wt.}$$

EXAMPLES. XXXVII.

1. If there are 9 pulleys altogether and each pulley weighs 1 lb., what force is required to support a weight of 104 lbs.?

2. What weight can be supported if there are 4 pulleys in the lower block and the total number be even, the weight of the lower block being 4 times the force P?

3. If weights of 5 lbs. and 6 lbs. support weights of 73 lbs. and 88 lbs. respectively, what is the weight of the lower block, and how many pulleys are there in it?

4. If the lower block weigh 24 lbs and contain 5 pulleys, the string being fastened to the lower block, what weight can be raised by a force equal to 21 lbs. wt.?

5. Find the pressure on the ground if a man with 8 weightless pulleys sustains a weight $\frac{1}{4}$ of his own.

6. Ten weights each of 20 lbs. are to be lifted to a height of 8 feet from the ground. Show how a system of pulleys may be arranged to lift them together, the weight of the pulleys being neglected, by exerting a force equal to one of them.

7. A man whose weight is 12 stone raises 3 cwt. by means of a system of pulleys, there being 4 pulleys in each block, and the string being attached to the upper block. What is his pressure on the ground?

8. What must be the relation between the radii of the pulleys in the lower block in order that they may be all grooved in the same piece?

159. Two other arrangements are usually described.

The figures show arrangements of three pulleys, there being as many strings as pulleys.

In I. each string is fastened to the supporting beam, in II. each string is fastened to the beam which supports the weight.

Notice that one figure is the *inversion* of the other.

W is the weight supported, the weights of the pulleys are neglected, the force applied is P.

CASE I.

The tensions of the strings being T_1, T_2 and T_3, we have since the tension of a string is the same throughout

$$T_1 = P.$$

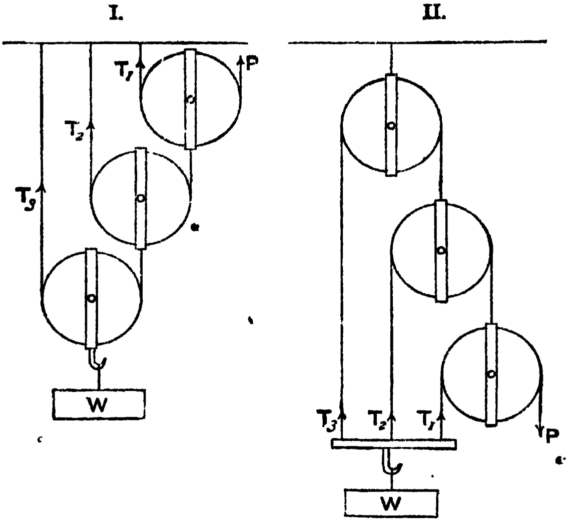

Fig. 119.

the tensions of the strings of the next pulley keep it in equilibrium, hence since they are parallel, by Art. 89

$$T_2 = 2T_1 = 2P,$$

similarly $$T_3 = 2T_2 = 2^2P,$$

also $$W = 2T_3 = 2^3P.$$

Thus $W = 8P$ is the required relation between W and P for equilibrium. The mechanical advantage is 8.

If there are n pulleys we see similarly that the relation would be

$$W = 2^n P.$$

Case II.

The tensions being T_1, T_2 and T_3, we have

$$T_1 = P,$$
$$T_2 = 2T_1 = 2P,$$
$$T_3 = 2T_2 = 2^2P.$$

And

$$W = T_1 + T_2 + T_3$$
$$= P + 2P + 2^2P = P(1 + 2 + 2^2)$$
$$= P(2^3 - 1).$$

Hence in this case the relation between W and P is

$$W=(2^3-1)P=7P.$$

For n pulleys the relation would be

$$W=(2^n-1)P.$$

160. If we take the weights of the pulleys into account, the previous equations are replaced by the following; w_1, w_2, and w_3 being the weights of the pulleys.

CASE I. $$T_1=P,$$

the first pulley is in equilibrium under the forces T_2+w_1 downwards and $2T_1$ upwards, hence

$$T_2+w_1=2T_1, \quad \text{or} \quad T_2=2P-w_1,$$

similarly $$T_3+w_2=2T_2, \quad \text{or} \quad T_3=2(2P-w_1)-w_2,$$
$$=2^2P-2w_1-w_2.$$

Also $$W+w_3=2T_3, \quad \text{or} \quad W=2(2^2P-2w_1-w_2)-w_3;$$
$$\therefore \quad W=2^3P-2^2w_1-2w_2-w_3.$$

If there are n pulleys we find similarly

$$W=2^nP-2^{n-1}w_1-2^{n-2}w_2\,\ldots\ldots-w_n.$$

When the weight of each pulley is w_1

$$W=2^nP-(2^n-1)w_1.$$

CASE II. $$T_1=P,$$
$$T_2=2T_1+w_1=2P+w_1,$$
$$T_3=2T_2+w_2=2(2P+w_1)+w_2,$$
$$=2^2P+2w_1+w_2.$$

Also $$W=T_1+T_2+T_3$$
$$=P+2P+w_1+2^2P+2w_1+w_2,$$
$$=P(1+2+2^2)+w_1(1+2)+w_2.$$
$$=7P+3w_1+w_2.$$

If there are n pulleys we find similarly

$$W=T_1+T_2+T_3+\ldots+T_n,$$
$$=P(1+2+\ldots+2^{n-1})+w_1(1+2+\ldots+2^{n-2})$$
$$+w_2(1+2+\ldots+2^{n-3})+\ldots+(2^2-1)w_{n-2}+(2-1)w_{n-1}.$$
$$=P(2^n-1)+w_1(2^{n-1}-1)+w_2(2^{n-2}-1)+\ldots$$
$$+(2^2-1)w_{n-2}+(2-1)w_{n-1}.$$

If the weight of each pulley is w_1 we have

$$W = P(2^n - 1) + w_1\{2^{n-1} + 2^{n-2} + \dots + 2 - (n-1)\}$$
$$= P(2^n - 1) + w_1(2^n - n - 1).$$

Notice that the weights of the pulleys in this case *assist* P.

161. The Pull on the Fixed Beam.

The fixed beam undergoes a downward pull which is clearly the resultant of the total weight supported and P.

In Case I. if R is this pull, we have

$$R = W - P,$$

in Case II. $$R = W + P.$$

If the pulleys have weight we must add the sum of their weights to W.

Ex. 1. There are 4 moveable pulleys arranged as in Class I., find W if P is equal to the weight of 10 lbs., neglecting the weights of the pulleys.

Here $$W = 2^4 \times 10 = 160 \text{ lbs. weight.}$$

Ex. 2. If in the last case the weight of each pulley is 4 ozs., find W.

In this case $$W = 2^4 \times 10 - (2^4 - 1)\tfrac{4}{16}$$
$$= 160 - \tfrac{15}{4}$$
$$= 156\tfrac{1}{4} \text{ lbs.}$$

Ex. 3. If the pulleys are arranged as in Class II., find W when P is the weight of 10 lbs., neglecting the weights of the pulleys.

Here $$W = (2^4 - 1)\,10 = 150 \text{ lbs.}$$

Ex. 4. Taking the weight of each pulley as 4 ozs., and P as 10 lbs., find W. In this case

$$W = (2^4 - 1)\,10 + (2^4 - 4 - 1)\tfrac{4}{16}$$
$$= 150 + 11 \times \tfrac{4}{16}$$
$$= 152\tfrac{3}{4} \text{ lbs.}$$

162. Motion of the Systems.

In the system of pulleys of Art. 157 if P descends a distance h the n strings in connexion with W are shortened a total amount h, hence each is shortened a distance $\frac{h}{n}$.

Hence if v is the velocity of P, the velocity of W is $\frac{1}{n}v$, therefore if f is the acceleration of P, the acceleration of W is $\frac{1}{n}f$.

For the motion of P we have the equation

$$\frac{P}{g}f = P - T \quad \text{.......................(i)},$$

and for the motion of W we have the equation

$$\frac{W}{g}\frac{f}{n} = nT - W \quad \text{.......................(ii)},$$

$\therefore$ from (i) and (ii),

$$\frac{f}{g}\left(nP + \frac{W}{n}\right) = nP - W,$$

or,

$$f = g\frac{n(nP - W)}{n^2P + W}.$$

In Case I., Art. 160, if the lowest pulley receive an upward displacement h, each portion of the string passing round it will be slackened. The next pulley must therefore be raised through a distance $2h$ to tighten the string, similarly, the next must be raised through twice the last distance, or 2^2h. If there are n pulleys the space moved through by the last will be $2^{n-1}h$. If the velocity of the lowest pulley is v, the velocities of the others are

$$2v,\ 2^2v,\ \ldots\ 2^{n-1}v,$$

and therefore if the acceleration of the lowest pulley is f, the accelerations of the others are

$$2f,\ 2^2f,\ \ldots\ 2^{n-1}f.$$

For a system of three pulleys we therefore have,

$$0 = T_1 - P,$$

$$\frac{w_1}{g}2^2f = 2T_1 - T_2 - w_1,$$

$$\frac{w_2}{g}2f = 2T_2 - T_3 - w_2,$$

$$\frac{W + w_3}{g}f = 2T_3 - (W + w_3).$$

From these equations T_1, T_2, T_3 and f can be found.

EXAMPLES. XXXVIII.

Exs. 1—8 come under Case I., Exs. 9—16 under Case II.

1. If the weight supported is 1 cwt. find the force applied, using 3 pulleys of Case I., neglecting their weights.

2. If the weight of each pulley in the last question be 2 lbs., find the applied force.

3. There are 4 pulleys, each weighing 2 lbs. What weight can be raised by a force equal to the weight of 20 lbs.?

4. There are three equal pulleys, and a force equal to the weight of $3\frac{1}{2}$ lbs. is required to support a weight of 21 lbs., find the weight of each pulley.

5. If there are 3 pulleys which weigh respectively P, $\frac{1}{2}P$, $\frac{1}{4}P$ beginning with the lowest, if a force P be applied, show that it can support a weight $5P$.

6. There are 5 pulleys whose respective weights are 5, 4, 3, 2, and 1 lbs., beginning with the lowest, what force will support a weight of 71 lbs.?

7. What is the mechanical advantage when there are n pulleys each as heavy as the weight to be raised?

8. By use of 4 pulleys of equal weight a certain weight can be supported by a force of 7 lbs. weight, but if a fifth similar pulley be used the same weight can be supported by a force of 4 lbs. weight. Find the supported weight and the weight of a pulley.

9. In the arrangement of Case II. find the weight supported by three pulleys, by use of a force of 10 lbs. weight, neglecting the weights of the pulleys.

10. If each pulley in the last question weighs 5 ozs., find the weight supported by a force equal to 10 lbs. weight.

11. If the pulleys are of different weights, show that the most advantageous arrangement is got by placing them in order of magnitude, the greatest being lowest.

12. If four pulleys each weighing 2 lbs. be used, find the force required to support a weight of 238 lbs.

13. Find what weight can be raised by a force of 7 lbs. weight, if there are 3 pulleys arranged in the least advantageous manner whose weights are 5, 4 and 3 lbs. respectivaly.

14. Find the mechanical advantage when the pulleys are five in number and the weight of each equal to $\frac{1}{8}$th of the applied force.

15. If a force P supports a weight W, show that a force $P+w$ would support a weight $W+w'$, where w is the weight of each pulley and w' is equal to $(2^n-1)w$.

16 In the case where 3 pulleys are used, if the diameter of each pulley be 4 inches, find to what point of the bar the weight should be attached in order that the bar may remain horizontal.

163. The inclined Plane.

An inclined plane is a plane making an angle less than a right angle with the horizon.

A line in the plane perpendicular to its intersection with the horizon is called a *line of greatest slope*, and a plane passing through the vertical and a line of greatest slope is called a *principal* plane.

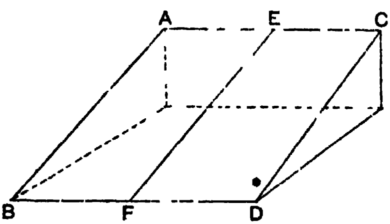

FIG. 120.

In the figure AB, CD and EF are lines of greatest slope.

The plane is supposed smooth, and hard enough to sustain any pressure.

164. A body is kept in equilibrium on an inclined plane: by the action of three forces, viz. its weight W, the pressure of the plane R (perpendicular to the plane), and a force P. Since W and R both lie in the same principal plane it is evident that that for equilibrium P must also lie in this plane.

We shall now take the two cases when P acts *horizontally* and *along the plane* respectively

CASE I., *P horizontal.*

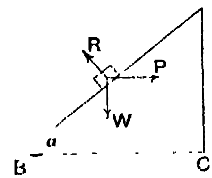

FIG. 121.

The sides CA, BC, AB of the triangle ABC are respectively perpendicular to the forces P, W and R, hence by the converse of the Triangle of Forces, Art. 50,

$$P : W : R = CA : BC : AB,$$

or

$$\frac{P}{CA} = \frac{W}{BC} = \frac{R}{AB}.$$

This may also be expressed

$$P : W : R = \text{height of plane} : \text{base} : \text{length}.$$

Since $P = W\dfrac{CA}{BC}$, it follows that $P = W \tan \alpha$,

$W = R\dfrac{BC}{AB}$, " " $W = R \cos \alpha$.

CASE II., *P along the plane.*

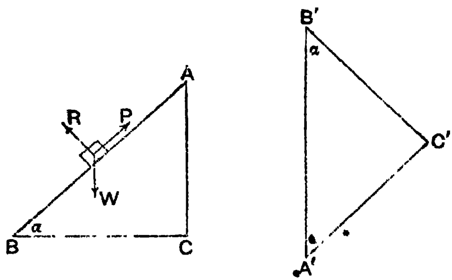

FIG. 122.

Make the vertical line $A'B' = AB$, also make the angles at A' and B' equal respectively to those at A and B; the

triangles ABC and $A'B'C'$ are then equal in all respects. Euc. I. 26.

Also the sides of $A'B'C'$ are parallel to the directions of P, R and W, viz.

$$B'A' \text{ to } W, \quad A'C' \text{ to } P, \quad C'B' \text{ to } R,$$

hence by the triangle of forces

$$\begin{aligned} P : R : W &= A'C' : B'C' : A'B' \\ &= AC : BC : AB. \end{aligned}$$

Or, $\quad P : R : W = \text{height} : \text{base} : \text{length}.$

Since $\quad P = W\dfrac{AC}{AB}, \quad \therefore P \text{ equals } W \sin \alpha,$

$$R = W\frac{BC}{AB}, \quad \therefore R \text{ equals } W \cos \alpha.$$

165. We may apply the method of work to Cases I. and II. as follows:

In Case I. let the body be displaced from the top to the bottom of the plane, then R does no work, and the total work done by W and by P must be zero, Art. 69, hence since the forces are in equilibrium

$$W \times AC - P \times BC = 0.$$

In Case II. take the same displacement, then similarly

$$W \times AC - P \times AB = 0.$$

Ex. 1. Find the weight which a force of 10 lbs. acting horizontally can support on an inclined plane whose height is 2 feet and base 13 feet.

We have $\quad \dfrac{W}{P} = \dfrac{\text{base}}{\text{height}},$

or $\quad W = 10 \times \frac{13}{2} = 65$ lbs. weight.

Ex. 2. What is the inclination of the plane to the horizon on which a horizontal force of $\sqrt{3}$ lbs. can support a weight of 3 lbs.?

$$\frac{\text{height}}{\text{base}} = \frac{P}{W} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}.$$

Thus the required angle is 30°.

Ex. 3. An inclined plane rises 2 in 5, what force acting along the plane will sustain a weight of 200 lbs.?

Here $\quad P = 200 \times \frac{2}{5} = 80$ lbs. weight.

EXAMPLES. XXXIX.

1. A weight of 8 lbs. is supported on an inclined plane by a horizontal force of 6 lbs. weight, if the length of the plane is 20 feet, what is its height?

2. A truck whose weight is 3 tons is kept at rest by a rope on a gradient of 1 in 60, find the tension of the rope.

3. What force acting horizontally could keep a weight of 12 lbs. at rest on a smooth inclined plane whose height is 3 feet and base 4 feet, and what is the pressure on the plane?

4. What horizontal force will keep in equilibrium a weight of 9 lbs. on an inclined plane and produce a pressure of 15 lbs. weight on the plane?

5. If the length of the plane is 40 inches and the height 8 inches, what is the mechanical advantage in the two cases?

6. In the case when the applied force acts horizontally the plane is turned over so that the height becomes the base, show that the force required to support a given weight will be greater if the original base exceeded the height.

7. What force must be applied at the circumference of a wheel 6 feet in diameter in order to drag a ton weight up a smooth inclined plane of 1 in 50 by means of a rope wound round an axle of 9 inches diameter?

8. If the pressure on the plane in Case II. be half the weight, what is the angle of the plane?

9. If the height of a plane be $3\frac{1}{2}$ feet and the length 12 feet, and a string can just bear a weight of $24\frac{1}{2}$ lbs. hanging freely, find the greatest weight it can support when fastened to a point on the plane.

10. If h and b are the height and base of an inclined plane, and if a force P can support a weight W when acting horizontally and a weight W' when acting along the plane, show that

$$W : W' :: b : \sqrt{h^2+b^2}.$$

11. If P be the horizontal force which can support a weight W and P' the force along the plane which is sufficient to support W, then

$$P : P' :: \text{length of plane} : \text{base of plane}.$$

166. When P makes an angle θ with the plane we proceed as follows:

The sum of the resolved parts of the forces along the plane is zero,

$$\therefore \quad P\cos\theta - W\sin a = 0 \;\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

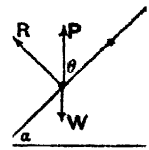

FIG. 123.

Also the sum of the resolved parts perpendicular to the plane is zero,

$$\therefore \quad P\sin\theta + R = W\cos a \;\ldots\ldots\ldots\ldots\text{(ii)},$$

or

$$R = W\cos a - P\sin\theta$$

$$= W\cos a - \frac{W\sin a}{\cos\theta}\sin\theta, \text{ from (i)}$$

$$= \frac{W}{\cos\theta}(\cos a\cos\theta - \sin a\sin\theta);$$

$$\therefore \quad R = \frac{W\cos(a+\theta)}{\cos\theta}.$$

Ex. 1. Show that the direction of the least force required to support a given weight on an inclined plane is along the plane.

From (i) if P is the force and θ its inclination to the plane

$$P\cos\theta = W\sin a,$$

$$\therefore \quad P = \frac{W\sin a}{\cos\theta},$$

we see that P is least when $\cos\theta$ is greatest, or when $\theta = 0$.

Ex. 2. A body whose weight is 20 lbs. is kept at rest on an inclined plane by a horizontal force of 10 lbs. together with a force of 10 lbs. acting up the plane. find the inclination of the plane to the horizon, and also the pressure on the plane.

Resolving along the plane we have

$$20\sin a = 10 + 10\cos a,$$

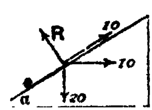

FIG. 124.

resolving perpendicular to the plane,

$$R = 20\cos a + 10\sin a.$$

From the first equation

$$\frac{\sin a}{1+\cos a} = \frac{1}{2}, \quad \text{or} \quad \frac{\sin^2 a}{(1+\cos a)^2} = \frac{1}{4},$$

hence

$$\frac{1-\cos a}{1+\cos a} = \frac{1}{4},$$

from which

$$\cos a = \frac{3}{5}.$$

It follows that

$$\sin a = \sqrt{1 - \frac{9}{25}} = \frac{4}{5},$$

hence

$$R = 20 \text{ lbs. weight}.$$

EXAMPLES. XL.

1. A body whose weight is 20 lbs. rests on an inclined plane whose inclination to the horizon is 60°, find the supporting force, it being supposed to make an angle of 30° with the normal.

2. A body weighing 6 lbs. is placed on a smooth plane which is inclined at an angle of 30°, find the two directions in which a force equal to the weight of the body may act to maintain equilibrium.

3. A weight $2P$ is kept in equilibrium on an inclined plane by a horizontal force P and a force P acting parallel to the plane; find the inclination of the plane to the horizon and the pressure on the plane.

4. If the pressure on the plane be an arithmetic mean between the weight and the applied force, and the inclination of this force to the horizon be $2a$, where a is the inclination of the plane,

$$\sin 2a = \tfrac{3}{4}.$$

5. A force P acting at an angle with the plane whose cosine is $\frac{3}{5}$ keeps a weight at rest. If P act at half its former inclination, find in what direction a force $\frac{5}{4}P$ must act in order to keep equilibrium.

167. The Screw.

The form of the screw is most easily described as follows:

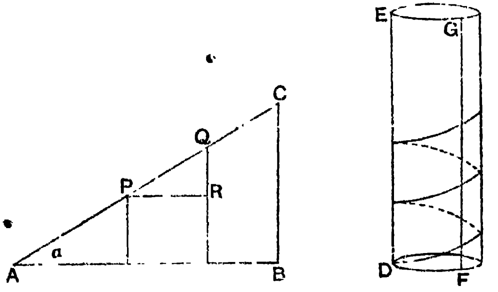

FIG. 125.

Take a right-angled triangle of paper ABC and a cylinder DE, then keeping BC parallel to the axis of the cylinder, wrap the base BA round the cylinder, the hypotenuse AC will form a spiral line on the cylinder.

Any line FG on the surface of the cylinder parallel to the axis will cut this spiral in a series of points. The distance between two successive points is called the *step* of the screw.

A broad groove is cut between the spirals leaving a ridge called the *thread.*

The screw works in a nut whose groove fits the thread. As the screw turns it also rises; the distance risen per unit angle turned through is called the *pitch.*

We see that when the screw has turned through four right angles it has risen a distance equal to a step, hence if p is the pitch,

$$\text{a step is equal to } p \times 2\pi.$$

If in the figure QR is a step, PR must be equal to the circumference of the cylinder, hence if r is the radius of the cylinder

$$\text{a step} = 2\pi r \tan a, \text{ where } a \text{ is the } \angle CAB.$$

Hence $$p \times 2\pi = 2\pi r \tan a,$$

or $$p = r \tan a.$$

Mechanical Advantage.

If a weight W is placed on the screw and the nut is held, the screw will descend, to support W a force P is applied at the end of a lever represented in the figure.

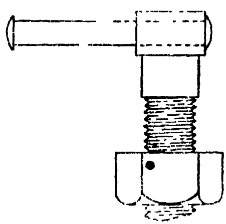

FIG. 126.

The condition of equilibrium is got from the principle of work as follows:

Let the screw descend through a step, then if P acts at a distance r' from the axis of the screw

$$W \times \text{step} = P \times 2\pi r',$$

or $$W \times p \times 2\pi = P \times 2\pi r',$$

hence $$Wp = Pr'.$$

168. Differential Machines.

The Differential Wheel and Axle.

This consists of three unequal cylinders having a common axis. Round the largest is coiled the rope by means of which the force is applied, round the other two the portions of the rope which support the weight.

We see from the figure that the ropes round the largest and smallest cylinders are coiled in the same manner, the rope round the middle cylinder in the opposite way to this.

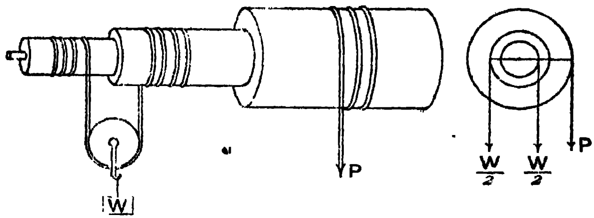

FIG. 127.

If W is the weight attached to the pulley, the tension of each rope is $\frac{W}{2}$. Let c, a, and b be the radii of the cylinders, beginning with the largest.

Taking moments about the axis, we have

$$Pc + \frac{W}{2}b = \frac{W}{2}a,$$

$$\therefore \quad \frac{W}{P} = \frac{2c}{a-b}.$$

By making a and b nearly equal we can get a large mechanical advantage.

Ex. In the differential wheel and axle, if the radius of the wheel be one foot and the radii of the two portions of the axle 5 and 4 inches respectively, what force will support a weight of 48 lbs.?

Ans. 2 lbs. weight.

The Differential Screw.

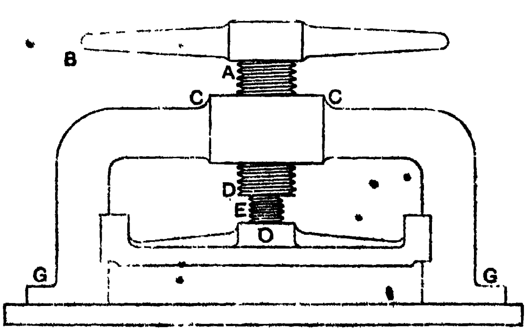

FIG. 128.

CG GC is a strong frame. In *CC* a groove is cut in which the thread of the screw on *AD* works, *AD* is hollow and the solid screw *DE* works inside it.

To the lower screw a board is attached which is guided to work up and down by smooth vertical grooves cut in *CG*, thus the lower screw can move up or down but cannot rotate.

The substance to which pressure is to be applied is placed beneath the board, W is the resistance offered by it.

Let l be the length of AB, p and p' the pitches of the respective screws. When AD has made a complete rotation it has descended a *step* or a distance $2\pi p$, in the same time the smaller screw has ascended a distance $2\pi p'$, *relatively to the larger one*, or *descended* a distance $2\pi (p-p')$, and this is the distance through which resistance is overcome, hence if P is the applied force we have by the principle of work

$$P 2\pi l = W 2\pi (p - p'),$$

or

$$\frac{W}{P} = \frac{l}{p-p'},$$

where l is the distauce AB from the axis at which P is applied.

The motion of the lower screw, corresponding to a considerable motion of AD, is extremely small.

The Differential Pulley.

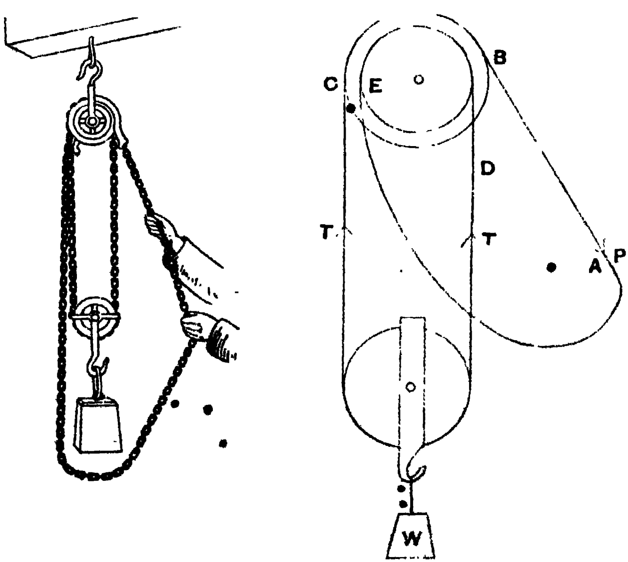

FIG. 129.

The figure on the right-hand side represents the principle of this machine, a sketch of which as used is given in the figure on the left.

An endless chain $ABCDE$ passes round the circumferences of two concentric wheels which are supplied with teeth.

It is found that for ordinary weights the chain from A to E may hang freely.

Let a and b be the radii of the larger and smaller wheels and T the tension of the chain which supports the weight W, then

$$\tfrac{1}{2} W = T.$$

Also by taking moments about the centre of the wheels, we have

$$Pa + Tb = Ta$$

$$\therefore \quad P = \tfrac{1}{2} W \frac{a-b}{a}.$$

EXAMPLES. XLI.

1. The arm of a screw-jack is one yard long and the screw has two threads to the inch. What force must be applied to the arm to sustain a weight of half a ton?

2. If a weight of 3 tons be raised 6 feet by a screw making 240 revolutions, find the force, the arm being 2 feet long.

3. A man by exerting a force of 12 lbs. with each hand can sustain a weight of 8 cwt. by means of a screw which has a double arm of 4 feet total length, find the pitch of the screw.

CHAPTER X.

FRICTION.

169. No surface is perfectly smooth, and when two surfaces are in contact the small ridges on one surface fit into small depressions on the other, so that the roughness of the surfaces retards the motion of one on the other.

This retarding force due to roughness is called the force of friction. This force is called into play in using all machines and part of the force applied is spent in overcoming friction, part only in doing useful work.

That friction is in many cases of practical advantage is seen from the fact that it is the friction between our feet and the ground which enables us to walk, and without friction we could not keep hold of objects. Another example of the use of friction is afforded by the locomotive engine. Take the case of an engine weighing 30 tons and suppose that the part of the weight borne by the driving-wheels is 10 tons. We shall see subsequently (Arts. 170, 171) that the force of friction is equal to the pressure × a number called the *coefficient of friction*, and since the coefficient of friction for wrought iron on wrought iron is ·2, the force of friction which acts on the driving-wheels of the engine is 10 × ·2 tons, *i.e.* 2 tons.

This friction opposes the rotation of the driving-wheels, *i.e.* acts in the direction in which the train is moving. It is thus the force which impels the train.

It is found that a pull of about 10 lbs. for every ton is sufficient to sustain motion, hence the engine should be able to draw along the level a weight of 4480 ÷ 10 tons, *i.e.* 448 tons.

170. Methods of Estimating Friction.

One method of determining the friction between two materials we shall now describe; a weighted slab of one material is placed on a horizontal planed surface of the other material, the slab is pulled by a horizontal string with gradually increasing force, more and more friction is thus called into play to counteract this force, until at length the slab begins to move *uniformly*. The friction then exerted is called *limiting* friction. The gradual increase of force is obtained by attaching a series of weights to the string.

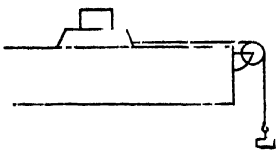

Fig. 130.

The weight required to make the slab move *uniformly* on the table is equal to the limiting friction, which can therefore be calculated.

As the result of experiment we thus arrive at the following law:—

If F is the limiting friction, and R the pressure of the slab on the plane (equal to the weight of the slab), then

$$F = \mu R,$$

where μ is the same for the two given materials whatever the value of R. The quantity μ is called the *coefficient of friction*. In the last Article the value of μ was $\cdot 2$.

It is necessary to give the slab a slight motion, since otherwise, owing to its weight, the surfaces become very slightly compressed and a force of coherence is introduced in addition to friction.

171. The Angle of Friction.

Another method of finding the value of μ is that of placing the weighted slab on a plane of the other material, and then tilting the plane until the slab begins to slide with a uniform motion down the plane. As before the slab should be *started*.

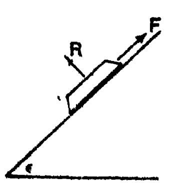

Fig. 131.

Let ϵ be the inclination of the plane for which the slab moves with

uniform velocity down the plane, then since the slab has no acceleration, by resolving along and perpendicular to the plane we obtain if F is the limiting friction

$$F - W\sin\epsilon = 0,$$
$$R - W\cos\epsilon = 0.$$

Therefore $$F = R\tan\epsilon.$$

Thus μ is equal to $\tan\epsilon$. The angle ϵ is called the *angle of friction.*

It is found that $F = a + \mu R$, where a is a small quantity independent of F and R gives more accurate results, thus for pine-wood on pine-wood it is found that the values of F (in lbs.) calculated from the formula

$$F = 1{\cdot}44 + 0{\cdot}252R,$$

give results differing from the actual values of F by only about ·3 lbs. The values of R for which this holds lie between 14 lbs. and 112 lbs. (Sir R. Ball's Experimental Mechanics.)

The formula $F = \mu R$ is usually sufficiently accurate for most purposes.

172. The Laws of Friction.

As a consequence of these experiments the following results have been established.

1. The greatest amount of friction is called into action when motion is about to take place, it is then called the *limiting friction.*

2. Limiting friction is proportional to the pressure, or

$$F = \mu R.$$

For different materials in contact μ has of course different values.

3. Friction is independent (i) of the extent of the area of contact, (ii) of the velocity with which one body moves over the other.

Since friction tends to prevent motion, not to originate it, it is often called a *passive resistance.* The above laws relate to *sliding* friction, when a body such as a wheel *rolls* on the ground *rolling* friction is called into play, this is very much less than sliding friction.

173. The Cone of Friction.

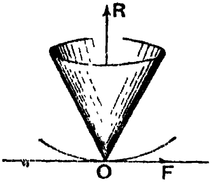

FIG. 132.

Take the case of any body touching a table at the point O, there is a normal force R and a force of friction F acting along the table, we have seen that the magnitude of F may be anything between zero and μR. The resultant of F and R is called the *total resistance* of the table. It is easy to see that $\frac{F}{R}$ is the tangent of the angle which the total resistance makes with the normal, hence if θ is this angle,

$$\tan\theta = \frac{F}{R}.$$

But we have seen that $\frac{F}{R}$ may have any value between zero and μ, *hence* $\tan\theta$ *must lie in value between* 0 *and* μ; that is between 0 and $\tan\epsilon$, where ϵ is the angle of friction, in other words

θ lies between 0 *and* ϵ.

So that the total reaction cannot make with the normal a larger angle than ϵ.

If we describe round the normal as axis a cone whose semi-vertical angle is ϵ, this is called the *cone of friction*, and we see that the direction of the resultant reaction lies within this cone. By increasing R we increase F, so that we may make F of any magnitude we please, but cannot bring its direction outside this cone.

This result is clearly true for all surfaces touching at one point.

174. Beam Resting against a Wall.

As an application of the preceding we shall take the case of a beam resting in a vertical plane against a rough vertical wall and the ground.

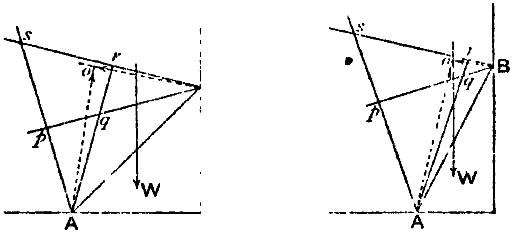

FIG. 133.

Let the vertical plane through the beam cut the cones of friction at A and B in the lines As, Ar, Bs, Bp. The total reaction at A must lie within the triangle Asr, and the total reaction at B must lie within the triangle Bsp. Hence their intersection is at some point O within the area $pqrs$. For equilibrium the line of action of the weight of the beam must pass through O, Art. 129. In the case of the first figure there cannot be equilibrium, in the case of the other, where the line of action of the weight does intersect the area $pqrs$, there will be equilibrium.

175. Body falling down a Rough Inclined Plane.

Another instance of the effect of friction is afforded by the case of a body falling down a rough plane whose inclination to the horizon is α.

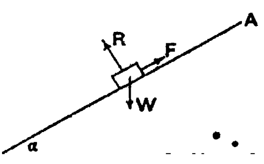

FIG. 134.

Since there is no acceleration perpendicular to the plane, we have

$$R - W \cos \alpha = 0.$$

The force down the plane is $W \sin \alpha - F$, where

$$F = \mu R.$$

Hence the force down the plane is

$$W \sin \alpha - \mu W \cos \alpha,$$

and the body's acceleration is

$$g(\sin \alpha - \mu \cos \alpha).$$

If v is the velocity with which the body reaches the bottom, supposing it to have started from rest at the top,

$$v^2 = 2g(\sin \alpha - \mu \cos \alpha) \times AB.$$

The square of the velocity is thus less than it would be if there were no friction by

$$2g\mu \cos \alpha \times AB.$$

Part of the work done by the weight of the body in falling has been used to generate its kinetic energy, the other part has been spent in overcoming friction, the energy corresponding to the work thus spent appears in the *generation of heat.*

176. Examples.

As illustrations of the laws of friction the solutions of several simple examples are added.

Ex. 1. A ladder rests against a wall and the ground, the coefficients of friction between the ladder and the wall and ground respectively are μ and μ_1. Find the inclination of the ladder when it is on the point of slipping down (it is then said to be in *limiting* equilibrium).

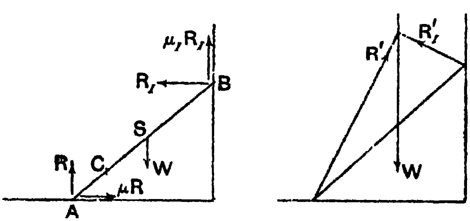

FIG. 135.

Resolving horizontally and vertically and taking moments about B we have, l being the length of the ladder and W its weight and θ its inclination to the horizon,

$$\mu R - R_1 = 0, \quad R + \mu_1 R_1 = W,$$

$$Rl \cos \theta = W\frac{l}{2} \cos \theta + \mu Rl \sin \theta.$$

From the first two equations,

$$R = \frac{W}{1+\mu\mu_1},$$

hence from the third equation

$$\cot\theta = \frac{\mu R}{R - \frac{W}{2}} = \frac{2\mu}{1-\mu\mu_1}.$$

Notice that the direction of the total reactions at A and B intersect on the line of action of W.

Ex. 2. If in the last case a weight w be placed on a rung of the ladder at C, where $BC = nl$, find the limiting position of equilibrium.

In this case the equations are

$$\mu R - R_1 = 0, \quad R + \mu_1 R_1 = W + w,$$

$$Rl\cos\theta = W\frac{l}{2}\cos\theta + wnl\cos\theta + \mu Rl\sin\theta.$$

Hence

$$R = \frac{W+w}{1+\mu\mu_1}, \ \cot\theta = \frac{\mu R}{R - \frac{W}{2} - nw} = \frac{2\mu(W+w)}{W(1-\mu\mu_1) - 2w(n\overline{\mu\mu_1+1}-1)}.$$

Observe that if $W(1-\mu\mu_1) = 2w(n\overline{\mu\mu_1+1}-1)$, θ is zero, or the ladder will rest in any position.

Ex. 3. Find the direction and magnitude of the least force required to drag a heavy body up a rough inclined plane.

The forces acting on the body are its weight W, the normal pressure of the plane R, the friction μR, and the required force P.

Under the action of these forces it is just on the point of moving up the plane. Hence these forces are just in equilibrium. Resolving along and parallel to the plane we have

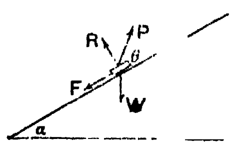

FIG. 136.

$$P\cos\theta = W\sin a + \mu R,$$

$$P\sin\theta = W\cos a - R.$$

Multiplying the last equation by μ and adding we get

$$P(\cos\theta + \mu\sin\theta) = W(\sin a + \mu\cos a).$$

Now put $\mu = \tan\epsilon$, then we have

$$P = W\frac{\sin a + \cos a\tan\epsilon}{\cos\theta + \sin\theta\tan\epsilon} = W\frac{\sin(a+\epsilon)}{\cos(\theta-\epsilon)}.$$

In order that P may be as small as possible $\cos(\theta-\epsilon)$ must be as large as possible, that is we must have $\theta = \epsilon$.

Hence the required force P makes with the plane the angle of friction and is of magnitude $W \sin(\alpha + \epsilon)$.

Ex. 4. A solid cube rests on a rough table, it is pulled by a horizontal string attached to a point in one face, it is required to determine whether the cube will slide or turn about its edge.

Let $ABCD$ be the section of the cube which contains the string.

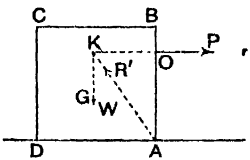

FIG. 137.

We suppose this section to bisect the cube and contain G its centre of gravity. If possible let the motion begin by the cube turning about its edge through A. The forces are the weight of the cube W, the tension of the string and the total reactions of the edge. All these total reactions may be replaced by one force R' which passes through A.

It is clear that R' must pass through the intersection K of W and P, and this it cannot do if the angle OAK is greater than ϵ.

Hence for turning about the edge $\frac{OK}{OA}$ must be less than $\tan \epsilon$,

that is, $\frac{OK}{OA} < \mu$, or $OA > \frac{1}{\mu} \times$ half the side of the cube.

Ex. 5. A drawer is to be pulled out by a single force parallel to the drawer, it is required to find how far from the middle of the end of the drawer the force can be applied so as not to cause the drawer to jam.

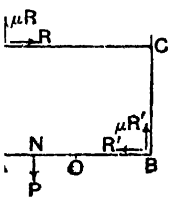

FIG. 138.

Let the point N be where P must be applied so that jamming may just begin. Let $ON = x$, where O is the centre of the end of the drawer whose length is l.

The drawer will now press against its sides at B and D, the total resistances at B and D meet on the line of action of P.

Since there is equilibrium, by resolving perpendicularly to the length of the drawer we see that $R - R' = 0$,

resolving parallel to it we get $P = \mu R + \mu R' = 2\mu R$,

taking moments about O $\quad Px = Rl$,

hence $$x = \frac{l}{2\mu}.$$

If P is applied at a distance from O greater than x the drawer will not move however great P may be. If the drawer is very long, or l great compared with AB, the point N may lie beyond A, *i.e.* the drawer will not jam at all.

EXAMPLES. XLII.

1. A weight of 60 lbs. is on the point of motion down a rough inclined plane when supported by a weight of $25\frac{1}{2}$ lbs. parallel to the plane, and on the point of motion up the plane when under a force of $32\frac{1}{2}$ lbs. parallel to the plane, find the coefficient of friction.

2. Show that the difference between the greatest and least forces which acting at an angle ϵ to a plane inclined at an angle a to the horizon sustain a weight W is $2W\frac{\cos(a+\epsilon)\sin 2\lambda}{\cos 2\epsilon + \cos 2\lambda}$, where λ is the angle of friction.

3. A ladder rests against a vertical wall and the ground, the coefficients of friction there being μ and μ'. If the ladder is on the point of slipping at both points, then if θ is the inclination of the ladder to the horizon,

$$2\tan\theta = \frac{1}{\mu'} - \mu.$$

4. A weight W is placed on a rough horizontal table and moved along by a weight P hanging over the edge and attached to W by an inextensible string, prove that the acceleration of W is

$$\frac{P-\mu W}{P+W}g,$$

where μ is the coefficient of friction.

5. A beam rests against a wall and the ground, the coefficient of friction at both ends being $\frac{1}{4}$, find when it is on the point of slipping.

6. A uniform beam standing on a horizontal floor and leaning against a vertical wall is just on the point of slipping when it is equally inclined to both floor and wall which are equally rough. What is the coefficient of friction, and what is the horizontal resistance of the wall?

7. Find the work done in dragging a load of 6 cwt. up a rough inclined plane whose height is 3 feet and base 20 feet, the coefficient of friction being $\frac{2}{15}$.

8. A homogeneous sphere cannot rest upon an inclined plane however rough.

9. A uniform ladder rests at an angle of 45° with the horizon with its upper extremity against a rough vertical wall and its lower extremity on the ground. If μ, μ' be the coefficients of friction between the ladder and the ground and wall respectively, show that the least horizontal force which will move the lower extremity towards the wall is

$$\frac{1}{2}W\frac{1+2\mu-\mu\mu'}{1-\mu'}.$$

10. A heavy cube (of weight 100 lbs.) rests on a rough floor on which it cannot slip. Prove that the least force required to begin to raise one edge of it off the floor is about $35\frac{1}{3}$ lbs., and find where it must be applied.

11. A man holds the end A of a uniform stick AB in his hand, the other end B being on the ground. If the stick be always kept at the same inclination (30°) to the horizon, and the angle of friction between B and the ground be 15°, the horizontal force required to *push* B with uniform velocity is to that required to *pull* it as $\sqrt{3}+1 : 2$.

12. Prove that a train going at the rate of 45 miles per hour will be brought to rest in about 378 yards by the brakes, supposing them to press with $\frac{5}{6}$ of the weight on the wheels of the engine and brake-van, which are $\frac{1}{2}$ of the weight of the train, the coefficient of friction being ·18.

13. A sphere of weight W is placed on a rough plane inclined to the horizon at an angle a which is less than the angle of friction, show that a weight $W\dfrac{\sin a}{\cos a - \sin a}$ fastened to the sphere at the upper end of a diameter parallel to the plane will just prevent the sphere from rolling down the plane.

14. Two rough spheres of equal radii but unequal weights W_1 and W_2 rest in a spherical bowl, their C.G.s coincide with their centres and the line joining them is horizontal and subtends an angle $2a$ at the centre of the bowl, prove that the coefficient of friction between them is not $< \dfrac{W_1 \sim W_2}{W_1 + W_2} \tan\left(\dfrac{\pi}{4} - \dfrac{a}{2}\right)$.

15. Weights W and W' of two different substances (coefficients of friction μ and μ') are supported on a rough double inclined plane (of angles a and a') by means of a string passing over the vertex. If the weight W be on the point of descending, prove that

$$\frac{W}{W'} = \frac{\sin a' + \mu' \cos a'}{\sin a - \mu \cos a}.$$

16. Find the least force which will sustain a weight of 10 lbs. on a plane rising 7 in 25, when the force acts along the plane and the coefficient of friction is $\frac{1}{4}$.

177. Effect of Friction on the Simple Machines.

We shall now investigate the effect of friction in the case of the simple machines considered in the last chapter.

Its effect is negligible in the case of the balance hung on knife-edges, but the friction at the axis becomes apparent in the case of the pulley and the wheel and axle, in the

case of the inclined plane and the screw the friction is considerable.

The pulley. The axle moves in a socket which it very nearly fits, thus the contact is at one point and the total reaction makes with the common normal to the axle and its socket the angle of friction.

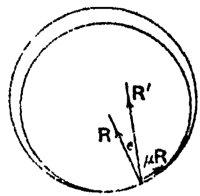

FIG. 139.

The direction of the total reaction is thus seen to touch a circle whose radius is $a \sin \epsilon$, where a is the radius of the axle.

To diminish the effect of friction it is usual to make the axle of a pulley as small as is consistent with strength and the radius of the pulley large, so that a comparatively small force applied at the circumference of the pulley may have the same *moment* as the friction on the axle.

The wheel and axle. If c is the radius of the axle on which the machine rests we have seen that R, the total resistance acts at a distance $c \sin \epsilon$ from its centre, also for equilibrium

$$R = P + Q.$$

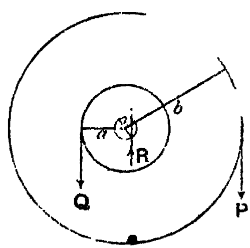

FIG. 140.

Hence taking moments about the centre of the axle,

$$Pb = Qa + (P + Q)\, c \sin \epsilon.$$

If there were no friction we should have

$$P_0 b = Qa.$$

The efficiency (Art. 138) is found by taking the ratio of P_0 to P, hence efficiency

$$= \frac{P_0}{P} = \frac{P_0 b}{Pb} = \frac{Qa}{Pb} = \frac{a}{b} \cdot \frac{b - c \sin \epsilon}{a + c \sin \epsilon}.$$

When ϵ is very small this is nearly equal to $1 - \frac{c}{ab}(a+b)\epsilon$.

178. The Inclined Plane.

Let P be the force acting at the angle θ with the plane and just large enough to drag a weight W up a rough inclined plane, then we have

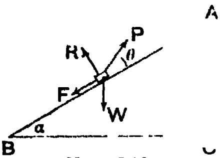

FIG. 141.

$$P\cos\theta = F + W\sin\alpha,$$

$$P\sin\theta = W\cos\alpha - R,$$

$$F = \mu R.$$

Whence

$$P(\cos\theta + \mu\sin\theta) = W(\sin\alpha + \mu\cos\alpha),$$

r $$P = W\frac{\sin\alpha + \mu\cos\alpha}{\cos\theta + \mu\sin\theta} = W\frac{\sin(\alpha+\epsilon)}{\cos(\theta-\epsilon)}.$$

If the inclination of the plane is less than the angle of friction, a force will be required to drag the body down the plane, if P' be this force making an angle ϕ with the plane

$$P'\cos\phi = F - W\sin\alpha,$$

$$P'\sin\phi = W\cos\alpha - R,$$

also $$F = \mu R;$$

$$\therefore\quad P'(\cos\phi + \mu\sin\phi) = W(\mu\cos\alpha - \sin\alpha),$$

$$P' = W\frac{\mu\cos\alpha - \sin\alpha}{\cos\phi + \mu\sin\phi} = W\frac{\sin(\epsilon-\alpha)}{\cos(\phi-\epsilon)}.$$

If θ is zero the force P acts along the plane, and the work done in dragging a body up the plane is

$$P \times AB,$$

also in this case $$P = F + W\sin\alpha,$$

$$R = W\cos\alpha,$$

$$\therefore\quad P = \mu W\cos\alpha + W\sin\alpha;$$

work done $$= P \times AB = \mu W AB\cos\alpha + WAB\sin\alpha$$

$$= \mu W \times BC + W \times AC.$$

Now $\mu W \times BC =$ work done in dragging the body along the base of the plane against friction.

$W \times AC =$ work done in raising the body against gravity through the height of the plane.

Hence the *total work done is = work done in dragging the body along the base considered rough + work done in raising it through the height of the plane.*

The screw. If the distance between two consecutive threads is h, and the mean radius is r, see Art. 167, we may regard the screw as an inclined plane of angle α, where

$$\tan\alpha = \frac{h}{2\pi r}.$$

Let P' be the force applied horizontally at the circumference of the screw in order to raise a weight W. Then by making θ equal to $-\alpha$ in Fig. 141, we obtain

$$P' = W\tan(\alpha + \epsilon).$$

If the effort P is applied horizontally at the end of an arm of length l, we have

$$P = \frac{r}{l} W \tan(\alpha + \epsilon).$$

Hence the mechanical advantage is given by

$$\frac{W}{P} = \frac{l}{r}\,\frac{1 - \tan\alpha\tan\epsilon}{\tan\alpha + \tan\epsilon} = \frac{l}{r}\,\frac{2\pi r - \mu h}{h + 2\mu\pi r}.$$

EXAMPLES. XLIII.

1. A body falls down a rough inclined plane of length l in the time t. If it be projected up the plane with the velocity with which it reached the ground, find the value of $\frac{l'}{l}$ and $\frac{t'}{t}$, where l' is the portion of l that it ascends and t' the time taken to do so.

2. Two equal weights rest on the faces of a double inclined plane whose angles are 30° and 60° and are connected by a string. If the weights are on the point of motion, show that the coefficient of friction is $2 - \sqrt{3}$.

3. A ladder 10 feet long weighing 42 lbs. and constructed with 9 steps dividing it into 10 equal spaces is placed against a vertical wall so as to make an angle a with the horizon, where $\sin a = \frac{4}{5}$.

If the coefficient of friction between the ladder and the ground and also between the ladder and the wall be $\frac{1}{2}$, show that a boy whose weight is 9 stone ascending the ladder will cause it to slip when he is stepping from the 8th to the 9th step.

4. A triangular plate ABC right-angled at B stands with BC on a rough horizontal plane. If the plane be gradually tilted round an axis in its own plane perpendicular to BC, the vertex B being downwards, prove that it will begin to slide or topple over according as the coefficient of friction is less or greater than $\tan A$.

5. A square board $ABCD$ stands on its base AD. If it be cut through along the diagonal AC, show that the least horizontal force which applied to AB will keep the triangle ABC from slipping is $\frac{1}{3}W$, W being the weight of the triangle, the coefficient of friction of the wood being $\frac{1}{2}$.

6. A heavy string rests on two given rough inclined planes of the same material passing over a smooth peg at their common vertex. If the string is on the point of slipping, show that the line joining its two ends is inclined to the horizon at the angle of friction.

7. A uniform bar is placed in a sloping position, its lower end on the ground its upper end in the air and supported by a smooth fixed peg against which it rests. If the ground is smooth show that it cannot rest in equilibrium. If the ground is rough (coefficient of friction μ), l the length of the bar and h the height of the peg from the ground, a the angle made with the horizon by the bar when on the point of slipping

$$\cos a \sin^2 a + \mu \sin a \cos^2 a = 2\frac{h}{l}\mu.$$

8. If in the wheel and axle the axle rests on rough bearings, the least force (acting downwards) that will raise a weight W is

$$\frac{b(1+\sin\lambda)}{a-b\sin\lambda}W.$$

9. Into a rough horizontal table at two points A and A' are inserted eyelet holes with slightly raised smooth edges. A weight of Q lbs. lies on the table midway between A and A', to it are attached strings the end of each string passing through an eyelet hole and sustaining a weight P. If Q be moved slowly along the table at right angles to AA', show that its greatest distance from AA' consistent with equilibrium is $\mu\frac{Q}{2\sqrt{4P^2-\mu^2Q^2}}AA'$, where μ is the coefficient of friction between Q and the table.

10. If a railway waggon weighs 6 tons and runs on 4 wheels each 9 feet in circumference, the axles being $1\frac{1}{2}$ inches in diameter; find approximately the number of foot-pounds of work done against the friction of the axles on their bearings whilst a mile is traversed if λ is the angle of friction for the bearings.

11. Three equal circular discs A, B and C are placed in contact with each other on a smooth horizontal plane, B and C being also in contact with a rough vertical wall. If the coefficient of friction between the discs and the wall is $2-\sqrt{3}$, show that there will be no motion when A is pushed directly towards the wall with any force.

12. A uniform rod AB is supported in a horizontal position with its extremity A in contact with the rough wall AC by the string CD. If AD and CD be $\frac{1}{3}$ and $\frac{2}{3}$ of AB respectively, prove that the coefficient of friction at A is $\frac{1}{\sqrt{3}}$.

13. A heavy particle is tied to one end of a light string of length $a\sqrt{2}$, the other end of which is fastened to a point distant a from a rough inclined plane of angle α, the angle of friction being λ $(\pi/4 > \alpha > \lambda)$. Prove that when the particle has its highest possible position on the plane the projection of the string on the plane makes with the line of greatest slope an angle $\sin^{-1}(\sin\lambda\cot\alpha) - \lambda$.

14. A heavy uniform rod lies on a rough horizontal plane and is acted on at one end by a force in the plane at right angles to its length. Show that as the force increases the rod will begin to rotate about a point dividing the rod approximately in the ratio 29 : 70.

15. A straight uniform beam 15 ft. long, weighing 60 lbs., is laid horizontally with its middle point upon a rough horizontal cylinder 3 feet radius, and is at right angles to the axis of the cylinder. If the angle of friction is 45°, find the greatest weight which can be placed upon one end of the beam without upsetting it.

CHAPTER XI.

MOTION IN A CURVED PATH.

179. Up to the present we have been concerned entirely with bodies moving in straight lines; we measured velocity by rate of change of distance along the line and acceleration, being time-rate of velocity, was also a vector quantity with its direction along the same fixed straight line. We have to examine now the motion of a particle which is made to describe a curved path lying in one plane.

Let fig. 142 represent the actual path of the particle described in the direction shown. If we measure a curved distance P_1P_2 and observe the time taken by the particle to cover this distance, we have at once the average velocity between these two points. Now suppose we take a series of positions for P_2 nearer and nearer to P_1; the average velocity thus calculated for each position of P_2 will be found to approach a limiting value when P_2 is very close to P_1 and this limiting value is defined as the velocity of the particle at P_1. The direction of this velocity v_1 is along the line P_1P_2 when P_2 is near P_1 and is thus ultimately along the tangent at P_1 to the path of the particle.

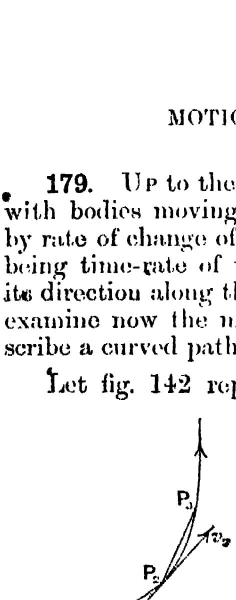

FIG. 142.

To determine the acceleration, we require a third point P_3; then if P_1, P_2, P_3 are consecutive neighbouring positions of the particle we can estimate the change of velocity in a short time. We see from the figure that the velocity changes not

only in size but in direction. Thus the acceleration of a particle moving in a curve is a vector whose direction does not coincide with the direction of the velocity. This is made clearer by the following graphical construction.

180. The Hodograph.

Let P_1, P_2, P_3 be consecutive positions of the particle in its path, and let v_1, v_2, v_3 be the velocities at these points.

From any base O draw Op_1, Op_2, Op_3 ... representing v_1, v_2, v_3 ... in magnitude and direction. If we have taken sufficient positions of the particle, the end points p_1, p_2 ... can be joined by a curve. This curve is called the Hodograph; the particle P moves along its path and the point p

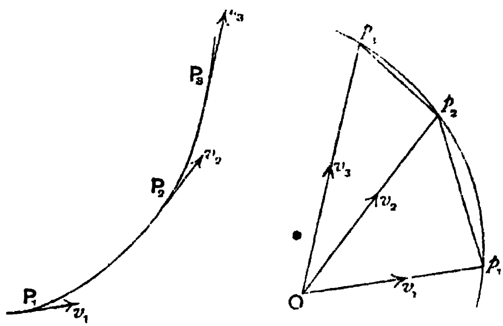

FIG. 143.

may be supposed to describe the hodograph. As the particle passes from P_1 to P_2 its velocity changes from v_1 to v_2; hence from the triangle of velocities, the chord p_1p_2 represents in magnitude and direction the velocity which must be compounded with v_1 to give v_2. Further while the particle moves from P_1 to P_2, the point on the hodograph moves from p_1 to p_2 in the same time. Hence the chord p_1p_2 divided by this time gives the average rate of change of velocity, that is, the average acceleration of the particle during the interval. Now suppose P_2 taken more and more closely after P_1; then the chord p_1p_2 becomes ultimately equal to the arc p_1p_2 and the rate of change of the

arc is the velocity of the point in the hodograph. Hence we have the theorem: the acceleration of the particle in its path is equal in magnitude and direction to the velocity of the point in the hodograph.

We shall consider now some special cases of curved paths.

Projectiles.

181. Suppose a heavy particle is projected from any point P with initial velocity V inclined at an angle α to the horizontal. Then if we suppose the motion to take place in a vacuum, or neglect the resistance of the air, the only force acting on the body during the motion is its weight; hence it has an acceleration g vertically downwards, while its horizontal velocity is unaltered throughout. Apart from gravity the particle would, after a time t, be at T, where $PT = Vt$; but on account of its vertical fall

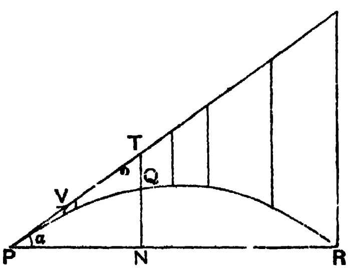

FIG. 144.

under gravity it must be at Q, where $TQ = \frac{1}{2}gt^2$. If we construct a large number of points Q in this manner, we can trace the path PQR as in the figure.

182. We may also consider the horizontal and vertical components separately. We have the following:

$$\text{Initially} \begin{cases} \text{Horizontal velocity} = V\cos\alpha, \\ \text{Vertical velocity} \quad = V\sin\alpha. \end{cases}$$

$$\text{After } t \text{ seconds} \begin{cases} \text{Horizontal velocity} = V\cos\alpha, \\ \text{Vertical velocity} \quad = V\sin\alpha - gt. \end{cases}$$

After the horizontal distance PN is $Vt \cos\alpha$, and the vertical rise QN is $Vt \sin\alpha - \frac{1}{2}gt^2$.

The particle will be at its highest point when its vertical velocity is zero, that is, after a time $\frac{V \sin\alpha}{g}$; consequently we have:

$$\text{time of flight} = \frac{2V \sin\alpha}{g},$$

$$\text{greatest height} = \frac{V^2 \sin^2\alpha}{2g},$$

$$\text{horizontal range} = \frac{2V^2 \sin\alpha \cos\alpha}{g}.$$

183. Consider the hodograph of this motion. The particle describes its curved path with velocity varying in magnitude and direction, while its acceleration is constant both in magnitude and direction. Hence the velocity of the corresponding point in the hodograph is constant in magnitude and direction; thus the hodograph is a vertical straight line described with uniform velocity numerically equal to g.

EXAMPLES. XLIV.

1. Find the horizontal and vertical spaces described in 3 seconds, when the components of the velocity of projection in those directions are 100 and 200 feet per second.

2. Find the greatest height to which a body will rise and its range, if it is projected with horizontal and vertical velocities of 400 and 800 feet per second.

3. The greatest height to which a body rises is 100 feet, find how far it will rise in 2 seconds.

4. The velocity with which a canon-ball leaves the gun has for its vertical and horizontal components velocities of $7\frac{1}{2}$ and 10 miles per minute, find its range.

5. Show that the direction of the velocity of the body in t seconds after projection makes an angle θ with the horizontal such that

$$\tan\theta = \frac{V \sin\alpha - gt}{V \cos\alpha}.$$

6. A particle is projected at an inclination θ to the horizon where $\cos\theta = \frac{4}{5}$, with a velocity of 1200 feet per second. Find the greatest height it attains and its range on a horizontal plane through the starting point.

7. A body is projected at an inclination α to the horizon, such that

$$\cos\alpha = \frac{1}{\sqrt{82}}$$

with a velocity of $\sqrt{82}$ feet per second. Show that after $\frac{1}{4}$ of a second its direction of motion is inclined at an angle of 45° to the horizon.

8. A particle is projected horizontally with a velocity of 30 ft./sec.; draw the graph of the path for the first 4 seconds.

184. The Greatest Range.

The horizontal range we have seen to be $\dfrac{V^2 \sin 2\alpha}{g}$.

For a *given* velocity of projection this will have the greatest value when $\sin 2\alpha = 1$, or $\alpha = 45°$.

Hence the greatest range for a given velocity of projection is got by projecting the body at an angle of 45° with the ground.

185. Range on an Inclined Plane.

A body is projected with a velocity V in a direction α from the foot of an inclined plane making an angle β with the horizon; it is required to find the range on the inclined plane, or how far up the plane the body will strike it.

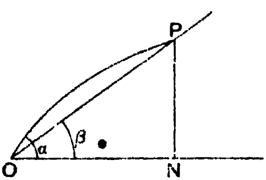

FIG. 145.

After t seconds the body will strike the plane at some point P, from P draw the vertical line PN.

Then

ON = horizontal space described in t seconds $= V\cos\alpha t$,

PN = vertical .... $= V\sin\alpha t - \frac{1}{2}gt^2$.

Also $\dfrac{PN}{ON} = \tan\beta$,

$$\therefore \quad \tan\beta = \frac{V\sin\alpha t - \frac{1}{2}gt^2}{V\cos\alpha t} = \tan\alpha - \frac{gt}{2V\cos\alpha}.$$

This determines t, giving

$$t = \frac{2V\cos\alpha}{g}(\tan\alpha - \tan\beta).$$

Thus

$$OP = ON\sec\beta = \frac{V\cos\alpha t}{\cos\beta} = \frac{2V^2\cos^2\alpha}{g\cos\beta}(\tan\alpha - \tan\beta).$$

Therefore the range is

$$\frac{2V^2\cos^2\alpha}{g\cos\beta}(\tan\alpha - \tan\beta)$$

$$= \frac{2V^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}.$$

Hence if V and β are given the range is greatest when

$$2\cos\alpha\sin(\alpha-\beta) \text{ is greatest,}$$

and $$2\cos\alpha\sin(\alpha-\beta) = \sin(2\alpha-\beta) - \sin\beta.$$

Thus for the greatest range $\sin(2\alpha-\beta)$ is greatest or

$$2\alpha - \beta = 90^\circ,$$

$$\alpha = \frac{\beta}{2} + 45^\circ.$$

186. The Path of a Projectile.

Let the particle be projected from P with velocity v in the direction PT.

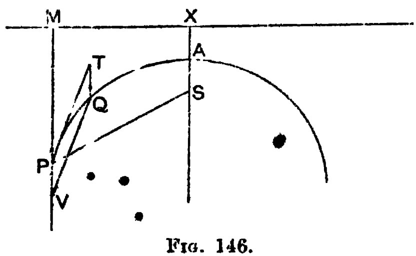

FIG. 146.

Then if Q is its position after time t, we have seen that

$$PT = vt; \quad TQ = \tfrac{1}{2}gt^2.$$

Let the vertical line PM be equal to $v^2/2g$, and draw MX horizontally through M. Obtain the point S by making $T\hat{P}S = T\hat{P}M$, and $PS = PM$; and complete the parallelogram $PVQT$.

Then if we describe a parabola with S as focus and MX as directrix, it follows from the construction that P is on the parabola, PT is tangent at P and PV is a diameter. Further we have

$$QV^2 = v^2t^2 = \frac{2v^2}{g} \cdot \frac{gt^2}{2} = 4SP \cdot PV.$$

Thus every position Q of the particle is on the parabola, which is consequently the path of the particle.

187. Velocity due to Fall from Directrix.

Since the point of projection P may be any point on the path and since

$$v^2 = 2gPM,$$

it follows from the previous construction that the velocity at any point is equal to that which it would have acquired by falling freely from the directrix.

EXAMPLES. XLV.

1. Find the direction and velocity with which to project a ball that it may pass horizontally over the top of a wall 50 yards off and 75 feet high.

2. A stone is projected into the air with a velocity of 200 feet per second in a direction inclined at 60° to the horizontal plane. With what velocity must another stone be projected vertically that the two stones may rise to the same height above the horizontal plane?

3. A boy with a stone aims at a mark 25 yards from him on a level 10 feet below his shoulder, with what velocity must he throw the stone horizontally so as to hit the mark?

4. A body is thrown from one extremity of the horizontal base of an isosceles triangle so as to pass just over the vertex and fall on the other extremity of the base. If α is the base angle and β the angle of projection show that $\tan\beta = 2\tan\alpha$.

5. Two particles projected with the same velocity from O pass through the same point P, prove that if α and β are the angles of projection

$$\alpha+\beta=\frac{\pi}{2}+i,$$

where i is the angle which OP makes with the horizon.

6. A shot whose mass is $\frac{1}{n}$th of the mass of the gun and carriage is fired at an inclination θ to the horizontal. If α be the inclination of the gun prove that

$$\tan\theta=\left(1+\frac{1}{n}\right)\tan\alpha.$$

7. A particle is projected at an angle $\tan^{-1}\frac{4}{5}$ to the horizon with a velocity of 200 ft./sec. Draw, to suitable scales, the path during the first 8 seconds, and also the hodograph.

8. From a point on a hill of inclination 30° one particle is projected up the hill and the other down with equal velocities, the angle of projection being in each case inclined to the horizontal at 45°. Show that the range of one particle is nearly $3\frac{1}{4}$ times that of the other.

9. Prove that 4 times the square of the number of seconds in the time of flight in the range on a horizontal plane is the height in feet of the highest point of the path.

10. A wet open umbrella is held with the handle upright and made to rotate round that handle at the rate of 14 revolutions in 33 seconds. If the rim of the umbrella be a circle of one yard diameter and its height above the ground 4 feet, prove that the drops shaken off the rim meet the ground in a circle of 5 feet diameter, the circumference of the rim being $\frac{66}{7}$ feet and the effect of the air being neglected.

11. Three bodies are projected simultaneously from the same point and in the same vertical plane, one vertically upwards, another at the elevation of 30°, and the third horizontally. If their velocities be in the ratio of $1:1:\sqrt{3}$ prove that the bodies will always be in the same straight line.

12. If the times taken by a projectile from P to Q and from Q to R are equal, its horizontal velocity being V, then if V_1, V_2 and V_3 are the velocities at P, Q and R respectively,

$$(V_3^2-V_1^2)^2=8(V_2^2-V^2)(V_3^2+V_1^2-2V_2^2).$$

13. A vertical line is divided into a number of equal parts

$$A_1A_2,\quad A_2A_3,\quad A_3A_4 \text{ etc.}$$

Show that if a particle be projected from O in the vertical plane through the line, OA_1, OA_2, &c. will meet its path in points such that the times of flight from each to the next are all the same.

14. A number of bodies are projected simultaneously from the same point O in the same vertical plane with different velocities such that if lines be drawn from O parallel and proportional to these velocities the extremities of these lines lie in a certain straight line AB. Prove that after a certain interval all the bodies will be situated in a straight line through O parallel to AB.

15. A bomb-shell on striking the ground burst, scattering its fragments with velocity V, find the area of ground covered by the fragments assuming that the shell falls on a horizontal plane.

16. Obtain a graphical construction for the angle of projection of a particle with given velocity in order to strike a given point.

17. For a given velocity of projection equal to $\sqrt{2gh}$, show that all possible paths touch an enveloping parabola with its focus at the point of projection and its vertex at a height h.

18. From a point on the ground at a distance x from the foot of a vertical column a ball is thrown at an angle of 45°, which just clears the top of the column and afterwards strikes the ground at a distance y on the other side; show that the height of the column is equal to

$$xy/(x+y).$$

19. Particles are projected simultaneously in perpendicular directions in the same vertical plane from two points in the same horizontal plane so as to strike this plane simultaneously at the same point after t seconds. Prove that the least distance between the points is gt^2 feet.

20. A particle is projected with a velocity of 56 f.s. and the range on a horizontal plane through the point of projection is 90 ft. Show that the difference of the possible times of flight is one second approximately.

21. Show that the distance from the point of projection of the furthest point on a horizontal plane, at a distance a below the point of projection, that can be reached by a projectile is $(a+2h)$, where the velocity of projection is $\sqrt{(2gh)}$.

22. From a point on the ground in front of a wall of height h stones are thrown with velocity V so as just to clear the top of the wall. Show that the furthest point of the top that can be thus reached is at a distance $(V^2-gh)/g$ from the point of projection. Find also the locus of the points where the stones strike the ground.

188. Uniform Motion in a Circle.

Consider the case of a body moving in the circumference of a circle with uniform velocity v. The rate at

which the radius joining the body to the centre turns round is called the body's *angular* velocity. This we shall denote by ω. Thus if θ be the angle turned through in a time t,

$$\frac{\theta}{t} = \omega.$$

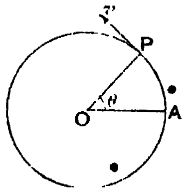

FIG. 147.

Hence if T be the whole time taken to describe the circumference

$$\frac{2\pi}{\omega} = T.$$

Also if an arc s is described in t seconds

$$v = \frac{s}{t} = a\frac{\theta}{t} = a\omega.$$

189. Acceleration in Circular Motion.

Let P and Q be successive positions of the body, then since its velocities at P and Q are equal in magnitude and perpendicular to OP and OQ respectively we may represent its velocity at P and Q by OP and OQ.

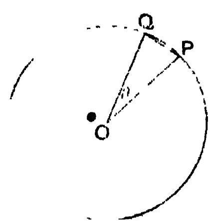

FIG. 148.

The velocity required to change the velocity at P to that at Q is therefore by the Triangle of Velocities represented by PQ and is perpendicular to PQ and directed *inwards*. Hence

$$\frac{\text{change of velocity in passing from } P \text{ to } Q}{\text{velocity at } P} = \frac{PQ}{OP} = 2\sin\frac{\theta}{2}$$

$$= \theta \text{ (when } \theta \text{ is very small)}.$$

Hence if τ denote the (very small) time occupied in going from P to Q,

$$\text{acceleration at } P = \frac{\text{change of velocity in passing from } P \text{ to } Q}{\tau}$$

$$= \frac{\theta}{\tau} \cdot \text{velocity at } P$$

$$= \omega v = \frac{v^2}{a} = a\omega^2.$$

Hence the measure of the acceleration at P is $\frac{v^2}{a}$ and is directed inwards along the radius. The body has thus no acceleration along the tangent at P.

190. We have seen that the acceleration of a particle moving uniformly in a circle of radius a is $\frac{v^2}{a}$, hence if its mass is m the force acting on the particle inwards along the radius is $m\frac{v^2}{a}$. This is the force which is required to keep the particle in its circular path; if, for instance, the particle is revolving on a smooth table at the end of a string attached to a fixed point, then the tension T of the string is the force which acts on the particle and keeps it in its path, therefore we must have

$$T = \frac{mv^2}{a}.$$

If the string breaks, the particle will move off along the tangent at the point at which it is when the string breaks, there being now no force to constrain it to move in a circle.

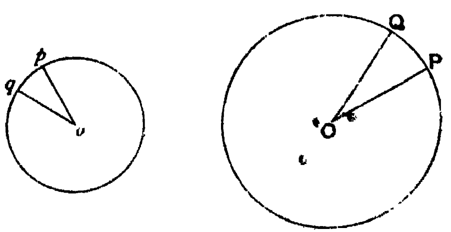

FIG. 149.

191. The hodograph of the path may be employed to find the acceleration of the body describing the circle, as follows:—

In fig. 149 let P, p and Q, q represent corresponding positions on the path and the hodograph.

The triangles POQ, pOq are similar, therefore

$$\frac{pq}{PQ} = \frac{Op}{OP} = \frac{v}{a}.$$

If t be the small time occupied in passing from P to Q, then since the chord PQ may be taken as being equal to its arc, therefore

$$\frac{PQ}{t} = v, \text{ and } \frac{pq}{t} = \text{acceleration at } P,$$

$$\therefore \frac{pq}{PQ} = \frac{\text{acceleration at } P}{v},$$

or from above $\dfrac{v}{a} = \dfrac{\text{acceleration at } P}{v}$,

thus the required acceleration is $\dfrac{v^2}{a}$, and acts parallel to pq and therefore parallel to PO.

192. Particle sliding on Smooth Curve under Gravity.

We have seen in Art. 60 that if a particle slides down a smooth inclined plane the increase in velocity is that which it would have gained by falling freely through the vertical height of the plane; this follows from the principle of work because the reaction of the plane is always at right angles to the path of the particle and thus does no work. Similarly if the particle moves on a perfectly smooth curve, the reaction R is always perpendicular to the direction

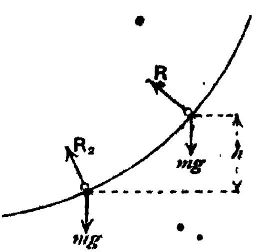

FIG. 150.

of motion; thus the only force which does work is the weight of the particle. Hence we have

$$v_2^2 - v_1^2 = 2gh.$$

193. Motion on a Vertical Circle.

Suppose a particle of mass m to move on the inside of a smooth vertical circle and to be projected initially with horizontal velocity u from the lowest point A. Let P be its position at any time. In one diagram we have the actual forces, R and mg, which act on the particle at the instant; in the other diagram we have the resultant forces

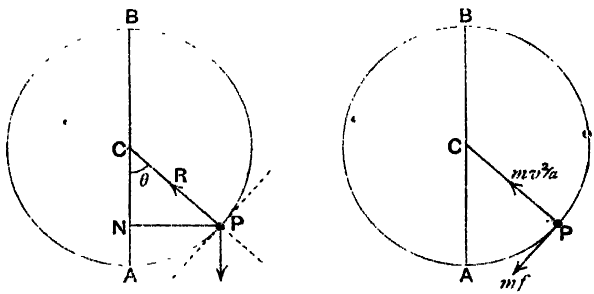

Fig. 151.

equal to (mass) × (acceleration). We know that if v is the velocity at P the acceleration inwards along the radius must be $\frac{v^2}{a}$; any other component acceleration must be perpendicular to this and we denote it by f along the tangent.

We equate the actual and effective forces of the two diagrams by resolving along the radius and tangent at P in each case; hence

$$mf = mg \sin \theta,$$

$$m \frac{v^2}{a} = R - mg \cos \theta.$$

But from the previous article,

$$u^2 - v^2 = 2g\,AN = 2ga\,(1 - \cos \theta).$$

Hence from the last two equations

$$R = mg\left(3\cos\theta - 2 + \frac{u^2}{ga}\right).$$

For the angle θ at which R becomes zero, the particle will leave the circle and proceed to describe a parabola under gravity. But if the velocity u is large enough the pressure R between the particle and the circle may be positive for all values of θ, so that the body describes the complete circle. In order that this may be so, R must be positive at the highest point B, that is when $\theta = \pi$; then the condition is

$$mg\left(-5 + \frac{u^2}{ga}\right) > 0,$$

$$u^2 > 5ga.$$

Thus the velocity of projection must be greater than that due to a fall through a height of $2\frac{1}{2}$ times the radius.

194. Conical Pendulum.

When a particle of mass m, attached by a string to a fixed point, moves uniformly in a horizontal circle whose centre is vertically below the fixed point the string and particle are said to form a conical pendulum.

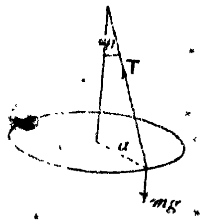

FIG. 152.

Let θ be the inclination of the string to the vertical, then since the weight of the particle is supported by the vertical component of the tension T of the string

$$T\cos\theta = mg \qquad \text{.....................(1)}.$$

To maintain the uniform circular motion of the particle a force $\frac{mv^2}{a}$ is required, where v is the velocity of the particle and a the radius of the circle it describes, therefore $T\sin\theta$ must be such that

$$T\sin\theta = \frac{mv^2}{a}.$$

If the particle makes n revolutions per second we have

$$v = 2\pi n a,$$

therefore $\qquad T \sin\theta = m 4\pi^2 n^2 a = 4m\pi^2 n^2 l \sin\theta,$

where l is the length of the string, or

$$T = 4m\pi^2 n^2 l \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Therefore from (1)

$$\cos\theta = \frac{g}{4\pi^2 n^2 l}.$$

We have now determined the tension of the string and the inclination of the string to the vertical.

EXAMPLES. XLVI.

1. A particle is placed on a rough horizontal plate ($\mu = \frac{3}{4}$) at a distance of 9 inches from a vertical axis about which the plate can rotate; find the greatest number of revolutions per second the plate can make without moving the particle.

2. At what number of turns per minute must a mass of 10 lbs. revolve horizontally at the end of a string 15 inches long so as to cause the same tension in the string as if one lb. were hanging vertically?

3. A particle slides on the outside of a smooth vertical circle, and is just displaced from the highest point. Find where it leaves the circle.

4. A heavy particle of mass m is moving on a smooth table in a circle being connected by a string, which passes through a hole in the table at the centre of the circle, with a particle of mass $2m$ which hangs vertically. What must be the velocity of the first particle?

5. A locomotive engine weighing 9 tons passes round a curve 600 feet in radius with a velocity of 10 miles an hour; what force tending towards the centre of the curve must be exerted by the rails?

6. The moon describes a circle of 60 times the earth's radius in 27·32 days. Taking the earth's radius as 3960 miles, show that gravity, reduced in proportion to the inverse square of the distance, is just enough to account for this.

7. A stone, attached to a string, is projected so as just to describe a vertical circle without the string becoming slack; show that the greatest tension in the string is six times the weight of the stone.

8. Show that, owing to the rotation of the earth, the apparent weight of a body is nearly one part in 289 less at the Equator than at a Pole.

9. A cannon weighing 12 cwt. hanging horizontally by two vertical suspending ropes at its ends projects a ball weighing 36 lbs. and is raised by the recoil 2·25 feet above its lowest position. Find the momentum and energy of the ball and of the cannon at the instant after discharge.

10. A particle weighing $\frac{1}{4}$ oz. rests on a horizontal disc and is attached by two strings 4 feet long to the extremities of a diameter. If the disc be made to revolve 100 times a minute about its centre, find the tension of each string.

11. When a train is travelling in a curve of 242 yards radius at 15 miles per hour, the string by which a heavy particle is attached to the roof of a carriage will be inclined to the vertical at $\cot^{-1} 48$.

12. Show that pieces of mud thrown from the top of a cab-wheel whose diameter is d feet, the cab moving with a velocity of v feet per second, will when they strike the ground, be at a distance $\frac{1}{4} v \sqrt{d}$ feet in front of the position then occupied by the point of contact of the wheel with the ground.

13. If T is the time of revolution of the bob of a conical pendulum at the bottom of a shaft of a mine of depth l, the pendulum being suspended from the surface of the Earth, the value of g at the bottom of the shaft is $\frac{4\pi^2 l}{T^2}\left(1-\frac{l}{a}\right)$, where a is length of the Earth's radius.

14. A particle is placed on the surface of a smooth sphere and slides down under the action of gravity; with what velocity will it leave the sphere if its initial angular distance from the highest point is α?

15. Two small bodies of 8 and 27 ozs. weight are laid on a smooth table and connected by a string 5 feet long passing through a small fixed ring in the table at the distance of 2 and 3 feet from the bodies and in the straight line joining them. They are then projected at right angles to the string towards the same side of it with velocities of 3 and 2 ft.-secs. respectively. Prove that each body will move in a circle and that if the string breaks after $\frac{3\pi}{13}$ seconds the bodies will meet at the end of the next second.

16. A particle revolves about a vertical axis, to which it is attached by a cord 8 inches long; if it keeps a stationary inclination at an angle of 27° to the vertical, find how many revolutions it is making per second.

17. If a uniform circular steel wire of small cross section is rotating in its own plane about its centre, show that the total action across a

section of the wire is mv^2, where m is the mass per unit length of the wire, and v is the linear velocity of any point of the wire. If the wire weighs 490 lbs. per cubic foot and can stand a strain of 90,000 lbs. per square inch, show that the greatest possible value of v is about 925 ft. per second.

18. A particle tied at the end of a string length a is projected horizontally from the lowest point of its orbit which is in a vertical plane through the point of suspension, with a velocity of $2\sqrt{ga}$. Where does it leave the circular path and where does it take it up again? Also find the velocity when it again passes the lowest point, and account for the loss of energy.

19. A particle of mass m lbs. is attached by a light string 2 ft. long to a fixed point A on a smooth horizontal table, and a particle of mass $2m$ lbs. is attached to the former particle by a light string 1 ft. long. The system revolves uniformly on the table making one revolution per second round A, the strings being stretched and in the same straight line. Find the tension of each string in pounds weight.

20. At a bend in a river the velocity of a certain part of the surface is 170 cm. per second and the radius of curvature of the lines of flow is 9,100 cm. Show that the slope of the surface in a section transverse to the lines of flow has a gradient of about 1 in 309.

21. A particle slides outside a fixed smooth circle in a vertical plane, starting at rest at the top, and another is projected from the lowest point, inside the circle, with just sufficient velocity to carry it to the top; prove that both leave the circle at the same point, and proceed to describe parts of the same parabola.

CHAPTER XII.

SIMPLE HARMONIC MOTION.

195. S.H.M. as Projection of Uniform Circular Motion.

Let a point P describe a circle of radius a with uniform angular velocity ω, and let T be the time of a complete revolution. If AOA' is any fixed diameter and PN is a perpendicular from P on to AOA', then as P revolves in the circle N moves back and forward along AOA'. The motion of N is said to be a simple harmonic motion.

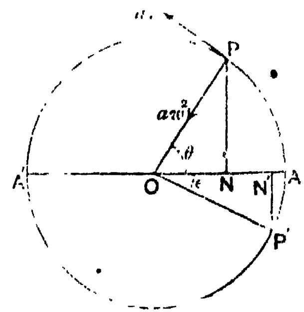

FIG. 153.

If θ is the angle between OA and OP at any instant, we have

$$x = ON = a \cos \theta.$$

Velocity of N in direction AOA' = component of velocity of P in direction AOA'

$$= a\omega \sin \theta = \omega \sqrt{a^2 - x^2}.$$

Acceleration of N in direction AOA' = component of acceleration of P

$$= a\omega^2 \cos \theta = \omega^2 x.$$

We see that the acceleration of N is directed always to a fixed point O and is directly proportional to the distance from O.

196. Period, Amplitude, and Phase.

As the point P makes a complete revolution of the circle from A, the point N moves from A to A' and back to A; then the motion is repeated in the same manner. Thus the time T is the *period* of the simple harmonic motion of N. Further, the greatest distance of N from the centre O is OA, or OA'; this distance a is called the *amplitude*. Suppose the time t is measured from the instant when P and N are at A; then at any subsequent time t we have

$$\theta = \omega t = \frac{2\pi t}{T}.$$

Hence $$x = a \cos \frac{2\pi t}{T}.$$

Velocity of N in direction $AOA' = \dfrac{2\pi a}{T} \sin \dfrac{2\pi t}{T}$.

Acceleration of N towards $O = \dfrac{4\pi^2}{T^2} a \cos \dfrac{2\pi t}{T} = \dfrac{4\pi^2}{T^2} x.$

We see that the period of a simple harmonic motion is independent of the amplitude, and from the last relation we obtain the important property

$$\frac{\text{acceleration}}{\text{displacement}} = \frac{4\pi^2}{T^2} = \text{constant ratio for all points.}$$

If we choose to measure the time from the instant when N is at some other point, say N', and P at the corresponding point P', we have

$$\theta = \omega t - \epsilon, \quad \text{where} \quad \epsilon = P\hat{O}A.$$

The angle ϵ is the *epoch*. The angle θ is called the *phase*.

If t is measured from the instant when N passes through O in the direction $A'OA$, we have $\epsilon = \frac{1}{2}\pi$, and

$$x = a \sin 2\pi t/T.$$

197. Production of S.H.M.

Let N be a particle of mass m. We have seen that its acceleration at any distance x from O is equal to $4\pi^2 x/T^2$

and is directed towards O; consequently, by Newton's second law, the resultant force on N must be equal to $\frac{4\pi^2}{T^2}mx$ in the direction of NO. Thus the force required to produce simple harmonic motion in the particle must be proportional to the displacement from a fixed point and directed towards it. The importance of simple harmonic motion lies in the fact that force of this kind is produced when elastic bodies are strained or deformed slightly from their natural state, so that simple harmonic vibrations about the equilibrium position are set up.

198. Weight suspended from Spiral Spring.

Suppose we have a spiral spring (or elastic string) whose natural unstretched length is a. It has been found, and can be verified, by experiment that if the spring is not extended too far, the tension is proportional to the extension; thus, if x is the whole length, the tension T is given by

$$T = (\text{constant}) \times (\text{extension})$$

$$= \frac{\lambda}{a}(x - a),$$

where λ is a constant to be determined for each spring.

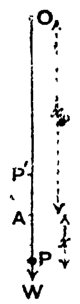

FIG. 154.

Let the spring be fastened at one end O and a weight W suspended at rest vertically by it; then the length x_0 of the spring is given by

$$W = \frac{\lambda}{a}(x_0 - a),$$

λ being the weight which would stretch the spring to twice its natural length, if it were sufficiently elastic to permit of this.

Now suppose the weight W to be set vibrating vertically, and let x be the further extension of the spring at any instant, so that the length of the spring is $x_0 + x$. The forces acting on the weight are the tension of the spring

upwards and W downwards, so that the resultant force tending to make x less is

$$\frac{\lambda}{a}(x_0 + x - a) - W = \frac{\lambda}{a}x.$$

Similarly if W is above its equilibrium position A, say at P', we find that the resultant force tending towards A is $\lambda x/a$.

Consequently the motion of W is a simple harmonic motion in a vertical line with A, the equilibrium position of W, as the centre of the motion.

If T is the period of the vibrations we have seen that the resultant force towards A must be

$$\frac{4\pi^2}{T^2}mx.$$

Hence $$\frac{4\pi^2}{T^2}m = \frac{\lambda}{a}.$$

The period T is given by

$$T = 2\pi\sqrt{\frac{ma}{\lambda}}.$$

199. The Simple Pendulum.

We shall consider now a case of simple harmonic motion in a curved path.

A weight W is suspended by a light inextensible string of length l, and oscillates under gravity in a vertical plane. Let θ be the angle the string makes with the vertical at any instant.

The bob P moves in an arc of a circle, and the tension of the string is always perpendicular to the path of P. Hence the component force on the bob in the direction of the tangent is $W \sin\theta$, or $W . PN/l$. Now if the oscillations are very small the straight line PN may be taken ultimately equal to the curved arc CP; thus the force on P is in the direction of the arc towards C and is directly proportional to CP. Hence the motion in the arc is simple

harmonic; or it is very approximately so when the oscillations are small enough. If then T is the period of the vibrations we have

$$\text{Force on bob along the arc} = \frac{mg}{l} \cdot CP = \frac{4\pi^2}{T^2} m \cdot CP.$$

Hence $$T = 2\pi \sqrt{\frac{l}{g}}.$$

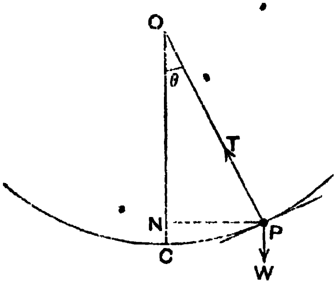

FIG. 155.

We can write down the velocity of the bob at P from the equation of work. Let v, v_0 be the velocities at P, C respectively; then since the tension T does no work we have

$$\tfrac{1}{2} m (v_0^2 - v^2) = mg\,CN,$$

$$v^2 = v_0^2 - 2gl\,(1 - \cos\theta).$$

We see that the period of the pendulum does not depend on the amplitude of the swings provided these are small enough; further the period varies with the length l and the value of g at the place, so that the simple pendulum may be used for determining g.

The time of a "beat" is half a complete period; thus the length l of a pendulum which beats seconds is given by

$$1 = \pi \sqrt{(l/g)}.$$

Taking g as 32, this gives l as 3·3 feet approximately.

EXAMPLES. XLVII.

1. A clock which gains 15 seconds a day has to be set right, find the alteration in the pendulum which should beat seconds.

2. The weight of 29·905 cubic inches of mercury in London is equal to that of 29·898 cubic inches in Manchester. How many seconds will a pendulum clock gain in a year in Manchester if properly regulated for London?

3. A pendulum whose length is l makes m oscillations in 24 hours. When its length is slightly altered it makes $m+n$ oscillations in 24 hours. Show that the diminution of the length is $\frac{2n}{m}l$ nearly.

4. Taking the values of g at the Equator and at the pole to be 32·09 and 32·25 respectively, find how much a clock regulated by a pendulum which would beat true seconds at the pole will lose in an hour at the Equator.

5. A mass of 5 lbs. hangs by a light spiral spring and makes three complete vertical oscillations of amplitude 2 inches in a second. Find the kinetic energy which it has when passing through its mean position.

6. A light elastic string is suspended from one end, when a heavy particle is gently attached to the lower end. Find how far the weight will descend and when it will first return to its original position.

7. An elastic string is stretched to double its natural length between two fixed points, and a particle of mass m fastened to its middle point. The particle is drawn aside towards one of the fixed points and then let go; find the time of a complete oscillation and also the greatest velocity acquired.

8. If, in S. H. M., c is the initial distance, v the initial velocity, and $2\pi/n$ the period, show that the time of reaching the centre is

$$\frac{1}{n}\tan^{-1}\frac{cn}{v}, \quad \text{or} \quad \frac{1}{n}\left(\pi - \tan^{-1}\frac{cn}{v}\right).$$

CHAPTER XIII.

IMPULSIVE FORCES AND IMPACT.

200. Impulse.

Given a mass m moving in a straight line under a constant force of F absolute units, u its velocity at the beginning, v its velocity at the end of an interval of time t, we know that

$$mv - mu = Ft,$$

or the momentum gained equals the total impulse of the force. If the change in velocity is observed to occur suddenly so that t is extremely small, then F must be very large, and it is impossible to measure either t or F separately; but the total effect is determined by the product of the two quantities, and is called an Impulse. Consequently an impulse is measured by the change of momentum produced. The unit of impulse is the same as the unit of momentum.

The effect can be illustrated by plotting a curve showing the change of momentum with the time; the interval of

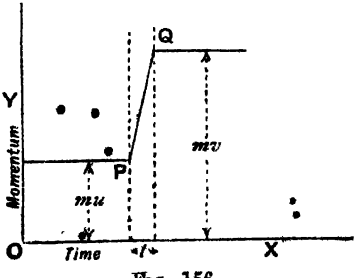

FIG. 156.

time between P and Q is supposed very small, hence PQ must be very steep—the slope being a measure of the acting force F.

Ex. 1. A body whose weight is 12 lbs. is made to increase its velocity from 30 to 40 miles an hour. Find the impulse.

$$I = \text{impulse} = 12\,(58\tfrac{2}{3} - 44) = 176\,\frac{\text{lb. ft.}}{\text{sec.}}\text{ units.}$$

Ex. 2. A particle, of mass 1 oz., is moving due E. at the rate of 3 ft. per sec.; what is the magnitude and direction of the blow that will cause it to move due N. at the same rate?

Ans. $\dfrac{3\sqrt{2}}{16}$ units of impulse, N.W.

201. Impact of Sphere on Fixed Wall.

Consider the case of a sphere impinging on a fixed smooth wall.

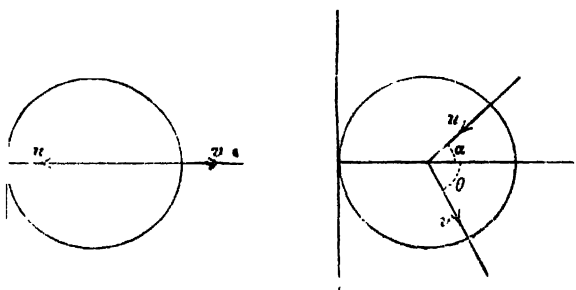

FIG. 157.

Firstly, let the sphere be moving directly towards the wall with velocity u. It has been found by experiment that it will rebound with a velocity v equal to eu, where e is a number less than unity called the *coefficient of restitution*; e does not depend upon u, but only on the materials of which the sphere and wall are composed.

By the impact momentum mu is destroyed and momentum emu is generated in the opposite direction, hence the impulse is measured by $mu\,(1 + e)$.

Secondly, let the velocity u before impact make an angle α with the normal to the wall, and let v, θ be the corresponding quantities just after the impact. Then since the surfaces are smooth there can be no action parallel to the face of the wall; hence

$$v \sin \theta = u \sin \alpha.$$

The component velocity at right angles to the wall is altered the same way as before, thus

$$v \cos \theta = eu \cos \alpha.$$

Hence we have $\qquad \tan \theta = \frac{1}{e} \tan \alpha.$

202. Geometrical construction for the Path after Impact.

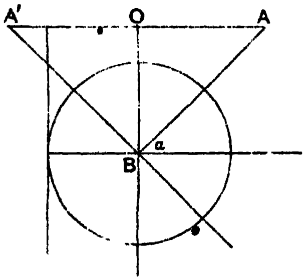

Fig. 158.

Let the centre of the sphere move towards the wall along the line AB, B being its position when it strikes the wall. Draw BO parallel and AOA' perpendicular to the wall, where

$$OA' = eOA.$$

Then

$$\tan \alpha = \frac{OB}{OA}, \quad \tan OA'B = \frac{OB}{OA'} = \frac{1}{e} \frac{OB}{OA}.$$

Therefore $OA'B$ is the angle θ of the last Article, and thus the ball will after impact move along $A'B$ produced.

If the sphere be a particle the point O will lie upon the wall.

In the foregoing we have supposed the motion to take place in a horizontal plane, but the result is easily seen to apply also in a vertical plane, giving the following result:

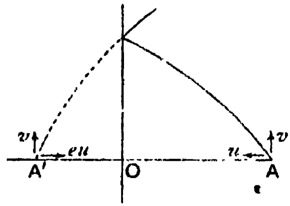

FIG. 159.

If a particle be projected from a point A with given horizontal and vertical components of velocity u and v, it will after impact at a smooth vertical wall describe the parabola which it would have described if it had been projected from A' with the velocities $-eu$ and v, where A, O, A' are points in the same horizontal line and such that

$$OA' = e \, . \, OA.$$

203. In the direct impact of a sphere with a fixed wall there will be an instant at which the sphere is momentarily at rest; this will be the instant of greatest compression. Let I_1 be the impulse of the elastic forces which have acted during the period of compression; then I_1 equals mu. After this instant the shape of the materials is being restored and the sphere finally leaves the wall with velocity eu; thus I_2, the impulse of the forces acting during the period of restitution, is equal to meu. Hence the coefficient e is the ratio of the impulses of the forces acting during restitution and compression.

204. Energy lost.

The loss of kinetic energy of the sphere is $\frac{1}{2}m(1-e^2)u^2$. This is only zero, if $e=1$, that is, for perfect elasticity; with imperfectly elastic substances some of the energy is dissipated in heating and other effects.

205. Impact of a Stream of Particles.

Suppose we have a stream whose area of cross section is S, composed of a very large number of small particles moving up to a fixed wall with velocity u. Let m be the mass of each particle, N the number of particles in unit volume of

the stream. If e is the coefficient of restitution, we have seen that the impulse on the wall due to an impact of a particle is $mu(1+e)$. Now in time t there are $SNut$ such

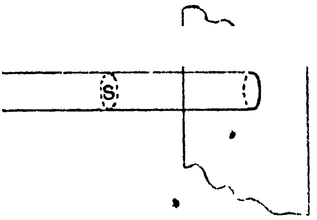

FIG. 160.

impacts, hence the total impulse on the wall in time t is $mNSu^2t(1+e)$. Let F be the average steady pressure per unit area on the wall whose total impulse Ft in the time t would equal that due to the impacts of the particles; then

$$F = mNu^2(1+e).$$

Thus we may regard the stream as exerting on the wall a steady pressure proportional to the square of the velocity, and to the density of the stream (since mN equals mass per unit volume).

206. According to the kinetic theory of gases, the pressure of a gas is due to the impacts on the walls of the vessel of the extremely large number of small particles of which a gas is supposed to consist. Suppose the gas is of density ρ and that the particles are moving about in all directions with an average velocity u; it can be shown that the pressure is proportional to the density ρ and to u^2. When the temperature is raised the average velocity u increases and hence the pressure becomes rapidly greater.

Ex. 1. A ball whose mass is 2 ozs. falls from a height of 64 feet and rebounds to a height of 25 feet. Find the value of e, and the impulse.

Ans. $\frac{5}{8}$; 13 lb.-ft./sec.

Ex. 2. A stream of water 1 inch in diameter is projected against a fixed wall with a velocity of 90 feet per second. If $e=\frac{1}{2}$, and a cubic foot of water weighs 1000 ozs. find the pressure on the wall.

Ans. 129·4 lbs. wt. approx.

207. Impact of Two Spheres.

Let two spheres of masses m, m' moving in the line of their centres come into collision; let u, u' be their velocities before impact, v, v' the values after impact, all measured in

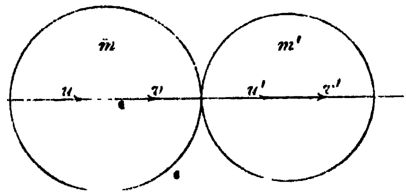

Fig. 161.

the same direction. By the laws of motion, the forces which act between them at impact act equally but in opposite directions on the two spheres. Hence the total momentum of the two spheres remains unaltered by the impact, remembering that momentum is a vector quantity, so that direction must be taken into account. Thus we can write down

$$mv + m'v' = mu + m'u' \text{.................} (1).$$

In order to find v and v' we need another equation, this is given by the experimental fact discovered by Newton that

the relative velocity of the spheres after impact is $-e$ times their relative velocity before impact, or

$$v' - v = -e(u' - u) \text{ ....} (2).$$

Solving these equations we obtain

$$v = \frac{(m - em')u + m'(1 + e)u'}{m + m'},$$

$$v' = \frac{m(1 + e)u + (m' - em)u'}{m + m'}.$$

Ex. 1. A ball whose mass is 4 lbs. moving with a velocity of 5 ft. per sec. impinges on a ball whose mass is 3 lbs. which is at rest. Find their velocities after impact, the coefficient of elasticity being equal to $\frac{1}{2}$.

In the present case equations (1) and (2) are

$$4v + 3v' = 20,$$

$$v' - v = \tfrac{1}{2} \times 5,$$

from which $v = \frac{25}{14}$, $v' = \frac{30}{7}$.

Ex. 2. Two balls whose masses are 5 and 6 lbs. respectively, directly meet each other. Before impact each is moving with a velocity of 2 ft. per sec. The coefficient of elasticity is $\frac{1}{3}$, find their velocities after impact.

Here $\qquad m=5,\ m'=6,\ u=2,\ u'=-2,$

hence $\qquad 5v+6v'=10-12=-2,$

$$v'-v=\tfrac{1}{3}(2+2)=\tfrac{4}{3},$$

from which $\qquad v=-\frac{10}{11},\ v'=\frac{14}{33}.$

Thus the velocity of each ball is reversed in direction.

208. Kinetic Energy is lost by Impact.

The fact that kinetic energy is lost in the impact of two spheres may be shown as follows; let E and E_1 be the total kinetic energy before and after impact respectively, then since by elementary algebra

$$(m+m')(mu^2+m'u'^2)=(mu+m'u')^2-2mm'uu'+mm'(u^2+u'^2),$$

therefore

$$2(m+m')E=(mu+m'u')^2+mm'(u-u')^2\ldots\ldots(3).$$

Similarly

$$2(m+m')E_1=(mv+m'v')^2+mm'(v-v')^2\ \ldots\ldots(4).$$

By subtracting and using (1), we have

$$2(m+m')(E-E_1)=mm'\{(u-u')^2-(v-v')^2\}.$$

Hence if $u-u'$ is numerically greater than $v-v'$, E is greater than E_1.

By use of (2) we see that

$$E-E_1=\frac{mm'}{2(m+m')}(u-u')^2(1-e^2),$$

and since e is less than unity $E-E_1$ is a positive quantity.

The kinetic energy which is apparently lost reappears in the form of vibrations of the molecules of the spheres.

209. Let I_1 be the impulse between the spheres up to the instant of greatest compression, and I_2 the impulse from that time until contact ceases.

At the instant of greatest compression the spheres are both moving with the same velocity which we may denote by V. Then since through I_1 the respective momenta of the bodies are changed from mu and $m'u'$ to mV and $m'V$,

$$I_1 = m(u - V) = m'(V - u').$$

Similarly $\quad I_2 = m(V - v) = m'(v' - V).$

Hence

$$I_1\left(\frac{1}{m} + \frac{1}{m'}\right) = u - u', \quad I_2\left(\frac{1}{m} + \frac{1}{m'}\right) = v' - v,$$

therefore $$\frac{I_2}{I_1} = \frac{v' - v}{u - u'} = e.$$

Ex. Two iron balls ($e = \cdot 66$), masses 5 and 4 lbs., meet directly with opposite velocities of 7 and 9 ft. per sec. respectively. Find the loss of kinetic energy. *Ans.* 5 ft.-lbs. approx.

210. Oblique Impact of Spheres.

When the spheres are not moving at impact in the direction of the line joining their centres, their respective velocities perpendicular to the line of centres are not altered by impact if the spheres are smooth, and their velocities resolved in the direction of the line of centres satisfy the two equations of Article 207.

Let u, u' be the velocities before impact, in directions making angles α, α' respectively with the line of centres.

Let v, v', β, β' be corresponding quantities after the impact; to determine these, we have the four equations

$$u \sin\alpha = v \sin\beta; \quad u' \sin\alpha' = v' \sin\beta';$$

$$mu\cos\alpha + m'u'\cos\alpha' = mv\cos\beta + m'v'\cos\beta';$$

$$v'\cos\beta' - v\cos\beta = -e(u'\cos\alpha' - u\cos\alpha).$$

211. Consider as another example of impact the action of a hammer or a pile-driver. Let M be the mass of the ram of the pile-driver and let it fall through a height H upon a pile whose mass is m. Suppose that the materials are inelastic, so that the ram remains in contact with the pile, while the latter is driven a distance h into the ground.

The velocity u of the hammer before impact is $\sqrt{(2gH)}$; if v is the common velocity of the hammer and pile immediately after the blow, the principle of conservation of momentum gives

$$Mu = (M + m)v,$$

$$v = \frac{M}{M+m}u = \frac{M}{M+m}\sqrt{2gH}.$$

For driving in the pile, we wish to have v as large a fraction of u as possible, so that it is advantageous to have M large compared with m. (Usually we neglect the mass of the pile compared with that of the ram, and we have v equal to u.)

The average resistance of the ground can now be calculated; let F be its measure in absolute units. During the motion, the ram and pile are acted upon by their weight downwards and by F vertically upwards; their initial velocity v downwards is destroyed in a distance h. Hence we have

$$F - (M+m)g = (M+m)\frac{v^2}{2h},$$

$$F = \left[1 + \left(\frac{M}{M+m}\right)^2 \frac{H}{h}\right](M+m)g.$$

EXAMPLES. XLVIII.

1. A sphere impinges directly on another sphere of double its mass moving with half its velocity. Show that if the coefficient of elasticity be $\frac{1}{2}$, the striking sphere will after the impact move with half its original velocity, and find the velocity of the other sphere.

2. A ball of mass M moving with velocity V impinges directly on a ball of mass m moving with a certain velocity, (i) in the same direction as M, (ii) in the opposite direction; if the coefficient of elasticity is $\frac{2}{3}$ and the subsequent velocity of m is four times as great in the former case as in the latter; show that the original momenta of the balls were in the ratio $1 - 2M/3m$.

3. A ball 5 ozs. in weight falls from a height of 20 feet upon a fixed horizontal plane, and on rebounding reaches a height of $11\frac{1}{4}$ feet; find the coefficient of elasticity and the measure of the impulse.

4. A cannon-ball strikes a smooth sea at an inclination of 1 in 100 and rebounds from the water at an equal inclination. Compare the blow with which it strikes the water with that required to stop it.

5. An elastic sphere let fall from a height of 16 feet above a fixed horizontal table will come to rest in 8 seconds after describing 65 feet, supposing the sphere to keep rebounding from the table with a coefficient of elasticity $\frac{7}{9}$.

6. A series of equal perfectly elastic balls are arranged in a straight line, one of them impinges directly on the next, and so on; prove that if their masses form a G.P. of which r is the common ratio the velocities with which they successively impinge will form a G.P. of which the common ratio is $\frac{2}{1+r}$.

7. A ball A of mass m impinges directly on another ball B of mass m' which is at rest. After the impact B impinges directly on a third ball C of mass m'' which is also at rest. If C has imparted to it the same velocity as A had at first, and all the balls are perfectly elastic

$$(m+m')(m'+m'')=4mm'.$$

8. Two equal marbles A and B lie in a smooth horizontal circular groove at opposite ends of a diameter. A is projected along the groove and after a time t impinges on B, show that a second impact will occur after a time $\frac{2t}{e}$.

9. A series of spherical balls of elasticity e are projected from a point and suffer reflexion at a smooth plane wall; show that their directions after reflexion pass through the same point.

10. A ball is projected vertically upwards with a velocity of 160 feet per second, and when it has reached its greatest height it is met in direct impact by another equal ball which has fallen through 64 feet; find the times from the instant of impact to that in which the balls reach the ground, their coefficient of elasticity being $\frac{1}{2}$.

11. A number of balls are dropped simultaneously from the heights m^2, n^2 &c. feet above a perfectly elastic plane, where m, n &c. are whole numbers. Taking g as 32 prove that they will all be in their original positions after $\frac{1}{2}M$ seconds, where M is the L.C.M. of m, n, &c.

12. A ball projected with velocity v at an inclination a will keep ricochetting from a smooth horizontal plane for a time $\frac{2v\sin a}{g(1-e)}$ and will have a range $\frac{v^2\sin 2a}{g(1-e)}$.

13. The velocity of a sphere moving on a smooth horizontal plane is reversed in direction after impinging successively on two fixed smooth vertical planes of the same material at right angles to each other.

14. A ball is projected from a point in a horizontal plane and makes one rebound, show that if the second range is equal to the greatest height which the ball attains, the angle of projection is $\tan^{-1} 4e$.

15. The diameters of each of two equal balls are two inches in length and the balls are moving in opposite directions each with velocity v with their centres on two parallel lines one inch apart. The coefficient of elasticity is $\frac{3}{4}$; find the velocity and direction of motion of each ball after impact.

16. A ball is projected from a point in the floor of a room of height h with velocity v and elevation θ in a vertical plane perpendicular to one of the walls so that $\sin\theta = \sqrt{\frac{2gh}{v^2}}$. After meeting one wall, the ceiling and the opposite wall it returns again to the floor. If the coefficient of elasticity be e, the distance between the walls a and the distance of the point of projection from the first wall d, prove that the distance from the second wall at which the ball meets the floor is

$$e^2(R - d) - ae,$$

where R is the horizontal range of a body projected with velocity v at an angle θ.

17. There are two equal perfectly elastic balls, one is at rest and is struck obliquely by the other, show that after impact their directions of motion are at right angles.

18. A mass of 5 cwt. falls 25 feet upon a pile weighing $1\frac{1}{4}$ cwt. and drives it 16 inches into the ground. Determine the resistance of the ground, supposed uniform.

19. Two inelastic spheres, m and m', are in contact, and m receives a blow through its centre in a direction making an angle a with the line of centres. Show that the kinetic energy generated is less than if m' had been absent in the ratio $m + m' \sin^2 a$ to $m + m'$.

20. A jet of water is directed from a circular nozzle 1 inch in diameter with a velocity which would carry it vertically 100 feet, so as to strike horizontally a wall at a height 50 ft. above the nozzle; prove that the pressure on the wall is that of about 48·4 lbs. wt.

21. $ABCD$ is a square formed by four smooth rods fixed on a smooth horizontal table. A particle is projected along the table from A at an angle $\tan^{-1}\frac{e+e^2}{1+e+e^2}$ with AB, e being the coefficient of restitution between the particle and each rod. Prove that the particle will, after impinging on BC, CD and DA, strike the corner B.

22. Two equal small spheres of steel are suspended by fine strings and constrained to swing in the same vertical plane as simple pendulums of equal length. In equilibrium they are just in contact. They are pulled apart in opposite directions until their centres are each 20 cm. from their equilibrium positions and are then simultaneously released. After the hundredth impact each is observed to swing so that its centre is 10 cm. from its equilibrium position. Calculate the coefficient of restitution for steel.

CHAPTER XIV.

GRAPHICAL STATICS. STRESSES IN RODS.

212. We have seen already how to solve graphically a simple problem in statics, namely when three forces acting at a point are in equilibrium. For we have only to draw a triangle such that the forces are parallel to its sides taken in order; then the three forces are in the same ratio as the lengths of the corresponding sides. We shall consider now the graphical solution of more complicated problems.

213. Construction for the Line of Action of the Resultant.

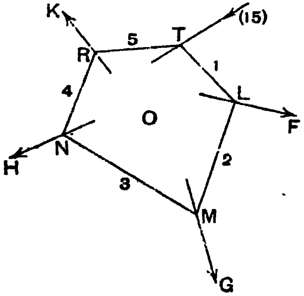

Fig. 162.

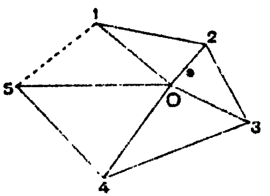

Fig. 163.

Let F, G, H, K be four forces acting in given lines in one plane on a rigid body; we require to find in magnitude, direction and position the resultant of these given forces.

Draw a line 12 representing the force F in magnitude and direction, having chosen a suitable scale; similarly draw 23, 34, 45 representing G, H, K respectively.

Case i. Suppose the point 5 so obtained does not coincide with the starting point 1, that is, the force polygon is unclosed. Then we know that the line 15 required to close the polygon represents the single resultant in size and direction. We have now to find the line of action of the resultant in the body, and for this it is sufficient to determine a single point on the line.

In fig. 163, take any point O, which we call the *pole* of the diagram, and join O to the points 1, 2, 3, 4, 5. Take any point L on the line of action of F in fig. 162, and draw LT, LM parallel to 01, 02 respectively; let M be the intersection of the latter line with the line of action of G.

Similarly draw MN parallel to 03, meeting H in N; draw NR parallel to 04, and finally RT parallel to 05. We shall show that the point T determined by the intersection of the first and last lines is on the line of action of the resultant force.

The figure $LMNRT$ is called a *funicular polygon* for the system of forces. The triangles of fig. 163 are triangles of force for the corners of the funicular polygon; hence we can replace

F by a force 10 in TL + a force 02 in LM,

G .... 20 in LM + ... 03 in MN,

H .... 30 in MN + ... 04 in NR,

K .... 40 in NR + ... 05 in RT.

Hence the four given forces are equivalent to a force 10 in TL together with a force 05 in RT; hence T is a point on the line of action of the resultant force, which is given in direction and magnitude by the line 15.

Case ii. *Force polygon closed.* Let there be five given forces F, G, H, K, P, where P is a force which is equal and opposite to the resultant of the other four and in a parallel line; then clearly the five forces are equivalent to a couple.

We proceed to draw the force polygon for the five forces; it will be as in fig. 163, and will in this case be a closed figure, the final point coinciding with the initial point.

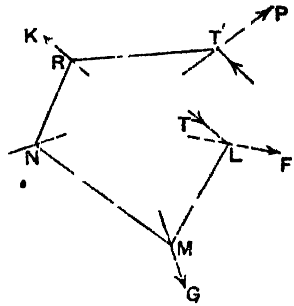

FIG. 164.

We next draw a funicular polygon as before, with LT' parallel to 01, LM to 02, ..., RT' parallel to 05, and finally $T''X$ to 01. Replacing the given forces by forces in the sides of the funicular polygon we see that we are left with a force 10 in TL and a force 01 in $T'X$; these are equal and opposite parallel forces. Hence if T'' does not coincide with T, that is, if the funicular polygon is unclosed, the system is equivalent to a couple. If both the force polygon and the funicular polygon are closed, the system is in equilibrium.

Observe the notation adopted in the above. For instance in figs. 162 and 163, if in the plane of the funicular polygon we attach numbers to the portions of the plane between the forces, the force F may be denoted by 12: it is given by the line 12 on the force diagram. Similarly when F is resolved into forces along TL and LM, they are denoted by 01 and 02 in both figures.

214. Parallel Forces.

The same method can be used when the forces are all parallel, as in the figures given below. We see that the force polygon in this case is the straight line 12345.

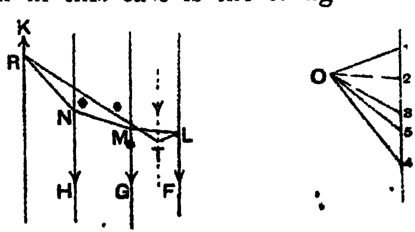

FIG. 165.

215. Frameworks.

When a number of rigid bars are joined together at their ends by smooth pins they are said to form a *frame*. We shall suppose for the present that the weights of the bars may be neglected.

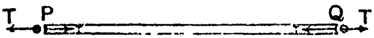

FIG. 166.

The following is an important proposition:

The action of a bar on a pin at its end acts along the bar.

For consider the bar PQ, it is in equilibrium under the actions of the pins at P and Q upon it, hence these actions must be equal and opposite and must therefore each act along PQ, let the magnitude of each of them be T, thus the action of the pin P on the bar PQ is T, and hence the action of PQ on the pin P is a force equal and opposite to T, *i.e. a force along the bar.*

These two equal and opposite forces of magnitude T acting on the bar are said to form the *stress* in the bar.

216. Examples.

We shall use the methods which have been described to investigate some simple cases.

I. A jointed frame PQR in the form of an equilateral triangle has a weight W attached to the joint P and the ends of its base, which is horizontal, rest upon fixed supports. Find the forces along the bars.

Letters are attached to the spaces divided from each other by the different forces, as explained in Art. 213.

Thus the weight W at P is denoted by AB, the upward force at Q which is equal to $\frac{1}{2}W$ by BC, the force with which the bar QR acts on the pin Q by CD, and so on.

Observe that in the case of any pin we take the forces acting on it in the contra-clockwise direction.

The force polygon consists here of a vertical line in which ab is of length W, bc and ca each of length $\frac{1}{2}W$.

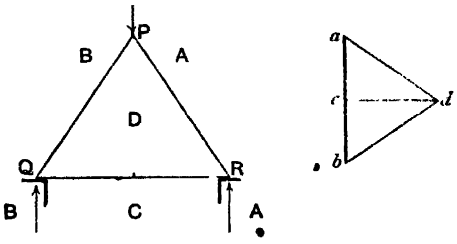

FIG. 167.

Through a and b draw ad and bd parallel respectively to the bars PR and PQ, join dc.

The pin P is in equilibrium under the action of the forces AB, BD and DA, but the sides of the triangle abd are parallel to these forces and ab is equal to the force AB, hence bd and da are equal respectively to the forces BD and DA.

The pin Q is in equilibrium under the forces DB, BC and CD, but the sides db and bc of the triangle dbc represent in magnitude and direction DB and BC, hence the side cd represents the force CD in magnitude and direction.

Hence cd must be parallel to the bar QR.

We saw that the action of the bar QP on the pin P is represented by bd, hence db is the action of the pin on the bar, that is the bar is subjected to a *compression*, such a bar is called a *strut.* The action of the bar QR on Q is represented by cd, and therefore dc represents the action of the pin on the bar, thus the bar is subjected to a *tension*, such a bar is called a *tie.*

II. A frame $PQRS$ has a weight W attached to the pin R and rests on two supports at P and S.

The supporting forces BC and CA at P and S are only given in direction, not in magnitude, we shall now find their magnitudes. Draw ab to represent W on some scale, the

forces BC and CA are represented by bc and ca but the position of c is not as yet known. Take any pole O and join Oa and Ob, we have to find the *direction* of the line Oc.

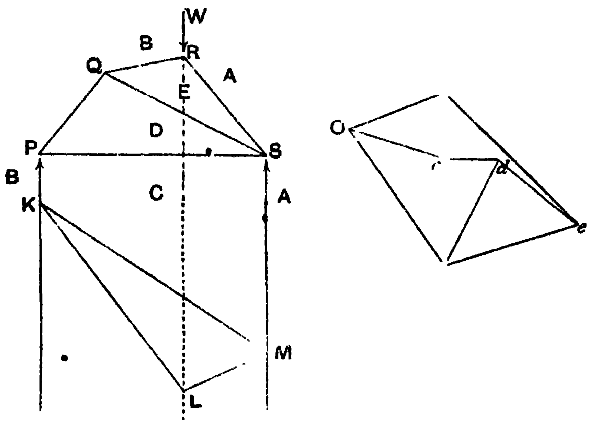

FIG. 168.

To do this we construct the funicular polygon, *i.e.* take any point K on the line of action of BC and draw KL parallel to Ob, through L draw LM parallel to Oa, we shall show that Oc is parallel to KM whose direction has been found.

By observing the force polygon we see that

$W(=ab)$ may be resolved into a force along $KL\,(=Ob)$ and a force along $ML\,(=aO)$,

$BC\,(=bc)$ may be resolved into a force along $LK\,(=bO)$ and a force in direction parallel to Oc,

$CA\,(=ca)$ may be resolved into a force along $LM\,(=Oa)$ and a force in direction parallel to cO.

The forces W, BC, CA are in equilibrium and their components along KL and ML destroy each other, hence the forces reduce to two, one through K and one through M, these two forces must be equal and opposite and therefore act along KM, and since we have seen that these forces are

each parallel to Oc, *therefore Oc is parallel to KM*. Hence c is found by drawing through O a line parallel to KM, and the forces BC and CA are represented by bc and ca.

The pin R is in equilibrium under the forces AB, BE and EA. Then by the same reasoning as in Ex. I. if we draw ae and be parallel to the bars RS and RQ we find that be and ea represent the forces BE and EA.

The pin Q is in equilibrium under the forces EB, BD and DE, but EB being represented by eb through b and e draw bd and ed parallel to the bars QR and QS, then as before we see that bd and de represent the forces BD and DE.

The forces which keep P in equilibrium are DB, BC and CD, join cd. then db and bc having been shown to represent DB and BC cd must represent CD; thus cd is horizontal.

The pin S is in equilibrium under the forces AE, ED, DC and CA which we have seen to be represented by ae, ed, dc and ca. The stresses in all the bars have now been found*.

Particular attention should be paid to the fact that the force denoted by any two of the large letters, *e.g.* BD, will be found in the other diagram by looking for the corresponding small letters, viz. in this case bd.

To find whether any particular bar, say QS, is a tie or a strut we proceed as follows:

Take one of its pins, say Q, the force on Q along QS is DE, now comparing with the line de in the other diagram we see that the *force on the pin* Q is from Q towards S, or the bar pulls the pin *in*, hence the pin pulls the bar *out*, thus QS is a tie.

III. A framework of five bars $PQRS$ has two opposite sides horizontal and to the pins at P and Q weights W and W' are attached, the frame resting upon fixed supports.

The lines ab and bc are drawn to represent W and W' respectively.

* That is, we can find their magnitudes by actually *measuring* the lines ae, be &c. These lines represent the forces in the scale in which ab represents W.

Any pole O is taken and joined to a, b and c; KL, LM and MN are drawn parallel to Oc, Ob and Oa, then as before, a line Od drawn parallel to KN will meet ab in d so that cd and da represent the forces CD and DA

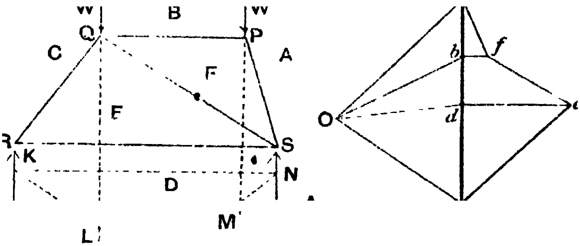

Fig. 169.

Through a and b, af and bf are drawn parallel to the forces AF and BF, and the forces on the pin P will be represented by ab, bf and fa. We then find the other forces as before.

IV. A symmetrical framework of the form shown in the figure is loaded at its highest point with a weight W and rests upon smooth supports. The diagram giving the forces is shown, the student will easily be able to follow the steps of the construction.

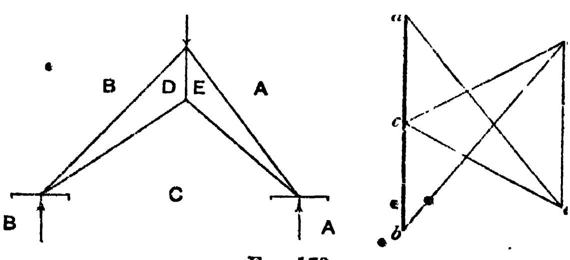

Fig. 170.

V. A system of bars consisting of two horizontal bars joined by cross bars equally inclined to the vertical is called a Warren girder. Equal weights are attached to the lowest

pins of such a girder and the system is supported at each end.

We find the forces on the pins in the order 1, 2, 3, 4, 5 as indicated in the figure. The forces NP, PC, CD at the

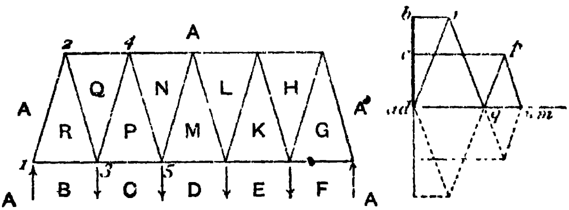

FIG. 171.

pin 5 are represented by np, pc and cd; in order to get the force MN we have to draw through d a line dm parallel to DM and through n a line nm parallel to MN, but these lines intersect in n, showing that n and m coincide, hence the force MN is zero and there is no stress in the corresponding bar.

217. Frameworks consisting of heavy jointed bars can be treated in a similar manner. Up to the present we have supposed the weights of the bars to be negligible compared with the loads supported by the framework; however we can take account approximately of the weight of the bars by supposing the weight of each bar to act in two equal parts at the ends of the bar.

218. Bending Moment and Shearing Force.

We wish to consider now more particularly the state of stress in a loaded beam.

Let AB be a light horizontal beam fixed at A and loaded at B with a weight W. Suppose the beam to be cut across at P, and consider the system of forces which must be supplied at P in order to maintain the part PB in equilibrium with W acting at B. Clearly, we must supply not only an upward force W at P but a couple G whose moment is equal to $W.BP$. But, when the beam is not cut at P, G and W must be the equivalent of the forces transmitted

across the section P of the beam; for the beam is slightly extended above and compressed beneath, so that the elastic forces tending to restore the natural state supply the system G and W at each section

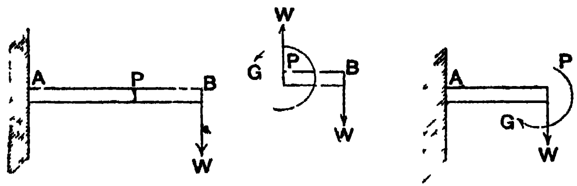

FIG 172.

Considered now as acting on the part AP of the beam the force W at right angles to the beam is called the Shearing Force, while the couple G equal to $W.PB$ is called the Bending Moment.

219. In general, for a beam AB in equilibrium under any given forces we can define the components of the action across any section P as follows:

(i) Pull or thrust in beam, equal to component along AB of the forces acting on either part of the rod, say the part BP.

(ii) Shearing Force, equal to the component of the same forces perpendicular to AB.

(iii) Bending Moment, equal to the moment of the forces on BP about the point P.

220. Beam freely Supported and Loaded at One Point.

Let AB be the beam loaded at C with a weight W, where $AC=a$, $BC=b$.

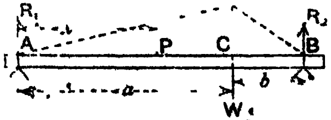

FIG 173.

Consider the bending moment at P, between A and C; it is equal to $R_1 x$, that is

$$bWx/(a+b)$$

Thus the B.M. increases uni-

formly from zero at A up to $abW/(a+b)$ at C, and similarly it decreases uniformly down to zero at B.

If we draw CD to represent the value at C, then the two straight lines AD, DB make up the B.M. diagram for the beam.

The Shearing force is numerically equal to $bW/(a+b)$ in the part AC and to $aW/(a+b)$ in the part CB.

221. Heavy Beam, Unloaded.

Let AB be a heavy beam of length $2l$ and of weight w per unit length. Consider a section P at a distance x from

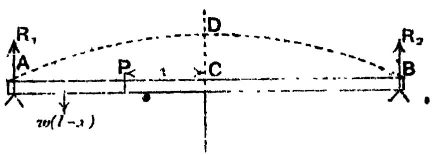

FIG. 174.

the middle point of the beam. The S.F. is clearly equal to $wl - w(l-x)$, or wx; thus it varies uniformly from $+wl$ at one end up to $-wl$ at the other end, passing through zero at the centre of the beam. The B.M. at P is equal to $wl(l-x) - \frac{1}{2}w(l-x)^2$, or $\frac{1}{2}w(l^2 - x^2)$.

Thus the B.M. increases from zero at each end up to its greatest value $\frac{1}{2}wl^2$ at C. The diagram of bending moment is a parabolic arc, and can be plotted along the beam as in ADB.

EXAMPLES. XLIX.

1. An equilateral framework ABC formed of light, jointed bars, is suspended by means of two vertical strings attached to A and B so that AB is horizontal. A weight of 15 lbs. hangs from C; find the stresses in the bars of the frame.

2. A horizontal beam AB, 18 ft. long, has loads 5, 11, 18 tons at distances of 6, 10, 15 feet from A. Find the supporting forces at the ends, and the shearing force and bending moment at sections of the beam 5, 9, 12 feet from A.

3. A uniform bar AB (weight 5 lbs., length 10 in.) is free to turn about the end A and is supported in a horizontal position by a cord BC attached to the end B and to a point C vertically over A, so that AC is 7 inches; and the bar carries a weight 5 lbs. at a point D such that BD is 3 inches. Draw a force diagram showing the tension of the cord BC and the reaction of the hinge at A, and prove that the latter reaction is about 9·4 lbs. wt.

4. Two heavy uniform rods AC, BC, of respective lengths 12 in. and 5 in. and masses 9 lbs. and 4 lbs., are freely hinged to one another at C and have their other ends attached to two smooth hinges fixed at points A and B, which are in a vertical straight line and 13 in. apart. Prove that the direction of the reaction of the hinge at C divides AB in the ratio 9 : 4; and draw a diagram for determining the reactions at A, B and C.

CHAPTER XV.

ENERGY OF ROTATING BODIES.

222. Moment of Inertia.

Suppose a rigid body is rotating about a fixed axis with angular velocity ω at any instant. Let the axis of rotation be through O at right angles to the plane of the figure; consider the motion of a small particle of the body of mass m at any point P whose perpendicular distance from the axis is r. By supposition the particle m is describing a circle of radius r with angular velocity ω, hence its velocity is $r\omega$, and its kinetic energy is $\frac{1}{2}mr^2\omega^2$. Now the kinetic energy of the whole body is the sum of the energies of all the particles of which it is composed; thus we denote the whole kinetic energy of the rotating body by

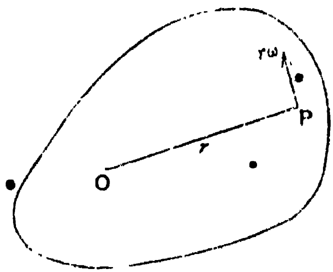

FIG. 175.

$$\Sigma\,(\tfrac{1}{2}mr^2\omega^2),$$

where the sign Σ means the summation of this expression for all the particles of the body. Now the angular velocity ω is the same for all the particles (since they make up a rigid body), hence we can write

$$\text{Kinetic energy} = \tfrac{1}{2}\omega^2 \,.\, \Sigma mr^2.$$

The factor Σmr^2 does not depend upon the velocity of rotation, thus it is a constant for a given body for a given axis of rotation; it is called the *moment of inertia* of the body for the given axis. Let M be the total mass of the body, then it is usual to write the moment of inertia I as

$$I = Mk^2,$$

where k is a length, and is called the radius of gyration; it is in fact the radius of the circle round which the whole mass might be supposed to be distributed in order to give the same energy of rotation as it actually possesses. We have then the result

$$\text{Kinetic energy of rotation} = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}Mk^2\omega^2.$$

223. Determination of Moment of Inertia.

Moments of inertia may be found experimentally in certain simple cases in the following way. Suppose a wheel is mounted so as to revolve on a horizontal axis, and a string is wound on the axle carrying a weight at its other end; the wheel is set in motion by allowing the weight to fall a certain distance before the string leaves the axle. If we can determine the angular velocity ω at this instant, the energy of the wheel is $\frac{1}{2}I\omega^2$; and if a is the radius of the axle, the kinetic energy of the falling mass is $\frac{1}{2}ma^2\omega^2$. This energy has been generated by the fall of m through a height h; then, neglecting friction, we have

$$\tfrac{1}{2}I\omega^2 + \tfrac{1}{2}ma^2\omega^2 = mgh.$$

Thus I can be determined.

In general, if the form of the body is known, I can be determined by calculation with the help of the Integral Calculus. We shall give here some of the more important results.

224. This table gives the moment for inertia in certain cases for axes through the centre of inertia of the body; by the following theorem we can obtain the moment of inertia about any parallel axis.

Table for k^2: Moment of Inertia $= Mk^2$

Thin rod of length $2l$ about an axis through its centre perpendicular to the rod	$\frac{l^2}{3}$
Thin rectangular plate of sides $2a$, $2b$	
(i) Axis through centre in plane of rectangle at right angles to the side $2a$	$\frac{a^2}{3}$
(ii) Axis through centre perpendicular to its plane	$\frac{a^2+b^2}{3}$
Rectangular block of sides $2a, 2b, 2c$, axis through the centre perpendicular to the side $2a$ $2b$	$\frac{a^2+b^2}{3}$
Thin circular disc of radius a	
(i) Axis a diameter	$\frac{a^2}{4}$
(ii) Axis through centre perpendicular to plane	$\frac{a^2}{2}$
Solid sphere about a diameter	$\frac{2a^2}{5}$
Solid cylinder, radius a, length $2l$	
(i) About its axis	$\frac{a^2}{2}$
(ii) About a line through its centre perpendicular to the axis of the cylinder	$\frac{a^2}{4}+\frac{l^2}{3}$

225. *The moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the centre of gravity G together with the moment of inertia of the whole mass placed at G about the original axis.*

Let G be the centre of gravity of the body; let the axis of rotation be through O perpendicular to the plane of the paper. Suppose the body divided into thin rods parallel to the axis of rotation; let one of them cut the plane of the figure at P and let m be its mass. Then since every part of the rod is at the same distance OP from the axis, its moment of inertia is mOP^2, and summing up for the whole body we have the moment of inertia about axis through $O = \Sigma m OP^2 = \Sigma m (OG^2 + GP^2 - 2OG \,.\, GN)$, where $PN =$ perp. from P on OG.

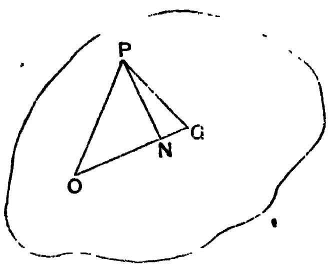

Fig. 176.

Now O and G are points fixed in the body; hence

$$\Sigma m OG^2 = OG^2 \Sigma m = M \,.\, OG^2.$$

Also G is the centre of gravity of the body, hence

$$\Sigma m OG \,.\, GN = OG \Sigma m GN = 0.$$

Hence $$\Sigma m OP^2 = \Sigma m GP^2 + M \,.\, OG^2,$$

which proves the theorem.

EXAMPLES. L.

1. The moment of inertia of a rod of length $2l$ about axis through one end perpendicular to its length $= \frac{1}{3} Ml^2 + Ml^2 = \frac{4}{3} Ml^2$.

2. The M.I. of a solid sphere of radius a about an axis at a distance c from the centre $=$ M.I. about a parallel diameter $+$ (Mass) c^2

$$= M\left(\tfrac{2}{5}a^2 + c^2\right).$$

3. Calculate the kinetic energy of a flywheel of 5 tons mass making 100 revolutions per minute (i) assuming the mass collected in a rim of 6 feet mean diameter, (ii) if the wheel is a disc of 6 feet diameter.

(i) The mass is all at the same distance from the axis, hence the moment of inertia $= (5 \times 2240 \times 3^2)$ lb.-ft.^{2} units.

Angular velocity in radians per second $= 2\pi$ (revs. per sec.)

$$= 6{\cdot}28 \times \tfrac{100}{60} = \tfrac{628}{60}.$$

$$\text{Kinetic energy} = \tfrac{1}{2} I\omega^2 = \tfrac{1}{2}(5 \times 2240 \times 9)\left(\frac{628}{60}\right)^2 \frac{\text{lb.-ft.}^2}{\text{sec}^2} \text{ units}$$

$$= \frac{5 \times 2240 \times 9 \times 628^2}{32 \cdot 2 \times 2 \times 60^2} \text{ foot-lbs.} = 171{,}360 \text{ foot lbs., approx.}$$

(ii) Moment of inertia $= \frac{1}{2}$ (mass) (radius)^2, hence the kinetic energy is one-half of the amount in the previous case.

226. Compound Pendulum.

If a heavy rigid body is swinging freely about a fixed horizontal axis, it is said to constitute a compound pendulum.

Let the axis of suspension pass through C and let G be the centre of gravity of the body.

Let ω be the angular velocity of the body when CG makes an angle θ with the vertical, and put $CG = h$.

Then if Mk^2 is the moment of inertia about a parallel axis through G, the moment of inertia about C is

$$M(k^2 + h^2).$$

Then the kinetic energy is equal to

$$\tfrac{1}{2} M (h^2 + k^2)\,\omega^2.$$

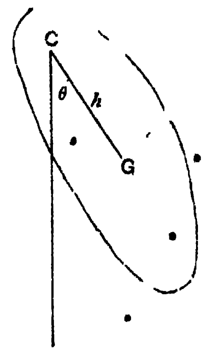

FIG. 177.

The only force which is doing work is the weight of the body, thus if ω_0 is the angular velocity when G is at its lowest point, the equation of work is

$$\tfrac{1}{2} M (h^2 + k^2)\,\omega_0^2 - \tfrac{1}{2} M (h^2 + k^2)\,\omega^2 = Mgh\,(1 - \cos\theta).$$

We compare this with the energy equation of a simple pendulum of length l with a bob of mass M, namely

$$\tfrac{1}{2} Ml^2\omega_0^2 - \tfrac{1}{2} Ml^2\omega^2 = Mgl\,(1 - \cos\theta).$$

We see that the two energy equations are of the same form and are identical if we put

$$l = \frac{h^2 + k^2}{h}$$

Consequently the heavy body swings back and forward like a simple pendulum of length $(h^2+k^2)/h$; and if the oscillations are small the period is independent of the amplitude and is given by

$$T = 2\pi\sqrt{\frac{h^2+k^2}{gh}}.$$

227. Atwood's Machine.

We shall consider what effect the inertia of the pulley has upon the motion of the masses in Atwood's machine.

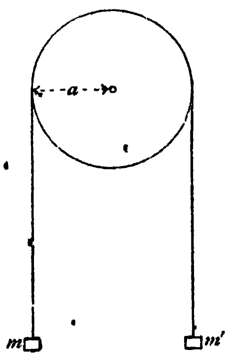

FIG. 178.

Let the pulley be of mass M and let its moment of inertia about the axis of rotation be Mk^2, and let its outer radius be a.

Then if at any instant u is the upward velocity of m', u will be the downward velocity of m; further, as we suppose the string not to slip on the pulley, the angular velocity ω of the pulley at the same time is given by

$$u = a\omega.$$

Hence we have

$$\text{Total kinetic energy} = \tfrac{1}{2}mu^2 + \tfrac{1}{2}m'u^2 + \tfrac{1}{2}Mk^2\omega^2$$
$$= \frac{1}{2}\left(m + m' + \frac{Mk^2}{a^2}\right)u^2.$$

If the machine starts from rest, and if the mass m has descended a distance x, the mass m' has risen the same distance and the work done by gravity is on the whole

$$mgx - m'gx = (m - m')gx.$$

But, assuming the bearings to be frictionless, gravity is the only force which does work or against which work is done. Hence the equation of work and energy gives

$$\frac{1}{2}\left(m + m' + \frac{Mk^2}{a^2}\right)u^2 = (m - m')gx.$$

If we write

$$m_1 = m + \tfrac{1}{2} M \frac{k^2}{a^2}; \quad m_2 = m' + \tfrac{1}{2} M \frac{k^2}{a^2},$$

this equation becomes

$$\tfrac{1}{2}(m_1 + m_2) u^2 = (m_1 - m_2) gx.$$

But this is the equation of energy for a simple Atwood's machine with masses m_1 and m_2 and a *massless* pulley. Hence the motion of the actual machine is the same as for a simple machine with the two masses increased each by the amount $\frac{1}{2}Mk^2/a^2$. The acceleration f of either of the masses m and m' is given by

$$f = \frac{m_1 - m_2}{m_1 + m_2} g = \frac{m - m'}{m + m' + M\frac{k^2}{a^2}} g.$$

EXAMPLES. LI.

1. A fly-wheel of 8 tons mass is running at 95 revolutions a minute. If the mass is supposed concentrated in the rim, whose inner and outer radii are 5 and 7 feet respectively, find the energy in foot-pounds.

2. A wheel is made to rotate by means of a weight of 6 lbs. hanging from a cord round the shaft which is 2 inches diameter. If the weight starts from rest and falls 10 feet in $1\frac{1}{2}$ minutes, calculate the moment of inertia of the wheel.

3. In an Atwood's machine, the weights are 200 grams and 230 grams, and the mass of the pulley is 50 grams. Find the acceleration if the pulley is a ring with spokes of negligible mass.

4. Find the time of a small oscillation of a rod 3 feet long suspended from one end.

5. Find the period of a square plate, one foot side, swinging in its own plane about a corner.

6. Show that the acceleration of a solid sphere rolling down a rough inclined plane of angle α without slipping is $\frac{5}{7}g \sin \alpha$.

7. If a supply of energy is worth one penny per horsepower-hour, what is the value of the kinetic energy of a wheel of mass 25,000 lbs. of mean radius 5 ft., when rotating at 100 revolutions per minute?

MISCELLANEOUS EXAMPLES.

1. Two trains, whose lengths were respectively 130 and 110 feet, moving in opposite directions on parallel rails were observed to be 4 secs. in completely passing each other, the velocity of the longer train being double that of the other. Find the rate of each train.

2. A ship sailing N.E through a current running 4 miles an hour after 2 hours' sailing has made good 4 miles S.E. Find the velocity of the ship and the direction of the current.

3. If the resistance of the air is always $\frac{1}{8}$ of the weight of a body in vacuo, find how high a body will go if shot up vertically with a velocity of 900 ft.-secs. Prove that the body will again reach the point of projection after 62·5 secs.

4. A carriage is slipped from a train moving 40 miles per hour. How far will it travel before coming to rest, reckoning the resistance of the rails $\frac{1}{27}$ of the weight?

5. When two unequal weights are connected by a string passing over a rough peg which has the effect of preventing motion until the tension of the string at one end be greater than that at the other by $\frac{1}{n}$th part of the latter tension, prove that the effect on the acceleration will be the same as if the peg were smooth and the smaller weight increased by $\frac{1}{n}$th part of itself.

6. There are n forces acting at O represented by $OA_1, OA_2, \ldots OA_n$. The middle point of A_1A_2 is joined to B_3, the middle point of OA_3, the middle point of the line thus drawn is joined to B_4, a point on OA_4, such that $OB_4 = \frac{1}{4}OA_4$, and so on. Prove that if P be the middle point of the last of all the lines thus drawn the resultant of all the forces is represented in magnitude and direction by $2^{n-1}OP$.

7. A number of smooth rods meet in a point A and rings slide down the rods starting from A. Prove that after a time t the rings are all on the surface of a sphere of radius $\frac{1}{4}gt^2$.

8. A body of mass M hanging vertically draws a body of mass M' up a smooth inclined plane by means of a string passing over a smooth pulley at the top of the plane. If M starts from the top of the plane, which is 14 feet from the ground, determine the velocity of M' just before M strikes the ground.

9. $ABCD$ is any quadrilateral and O is the intersection of two straight lines bisecting the opposite sides of the quadrilateral. Prove that forces acting at O represented by OA, OB, OC and OD are in equilibrium.

10. The sides AB, BC, CD and DA of the quadrilateral $ABCD$ are bisected at E, F, G and H respectively. Prove that the resultant of the two forces represented by EG and HF is represented in magnitude and direction by AC.

11. A ball weighing 12 lbs. leaves the mouth of a cannon horizontally with a velocity of 1000 feet per second; the gun and carriage, together weighing 12 cwt., slide on a smooth plane whose inclination to the horizon is 30°. Find the space of recoil up the plane, having given that the pressure caused by the explosion on the ball and on the end of the bore of the cannon is the same.

12. Four forces P, Q, R and S, no two of which are parallel, act in one plane. The resultant of P and Q meets that of R and S in A, the resultant of P and R meets that of Q and S in B, that of P and S meets that of Q and R in C; prove that A, B, C lie in one straight line.

13. In the triangle ABC the line DE is drawn parallel to the side BC and meeting the other sides in D and E, the lengths of DE and BC are b and a respectively. If h be the line drawn from A to bisect BC, prove that the distance of the c.g. of the figure $BCED$ from A is

$$\tfrac{2}{3}h\,\frac{a^2+ab+b^2}{a(a+b)}.$$

14. A heavy triangular plate lies on the ground. If a vertical force applied at the vertex A is just great enough to begin to lift that vertex off the ground, show that the same force will suffice if applied at B or C, the other vertices.

15. A fly-wheel is brought to rest after n revolutions by a constant frictional force applied tangentially to the circumference. If k be the kinetic energy of the wheel before the friction is applied and r its radius show that the force is $k/2\pi nr$.

16. If equal triangles be cut from the corners of a given triangle by lines parallel to the respective opposite sides, the c.g. of the remaining portion will coincide with that of the triangle.

17. Forces P, Q, R, S act in the sides AB, BC, CD and DA of a rectangle $ABCD$. Find the necessary and sufficient conditions that they should be equivalent to a single force acting at A.

18. $ABCD$ is a square, E the middle point of AB is joined to C and BD is drawn. Forces of 4 and 6 lbs.-wt. act respectively in AB and BC, and forces of 3 and 2 lbs.-wt. in AD, DC respectively, forces of $\sqrt{2}$ and $5\sqrt{5}$ lbs.-wt. act in BD and CE respectively, prove *by taking moments alone* that the system is in equilibrium.

19. $ABCDEF$ is a regular hexagon, and five forces each equal to P act along the straight lines joining A to the other vertices. Show that their resultant is $P(2+\sqrt{3})$.

20. Two bodies connected by means of a string passing over a smooth peg touch one another at one point, show that the stress between them cannot be horizontal unless their weights are equal.

21. A man sits in a chair which is suspended from the axle of a pulley; this pulley rides on a rope which is fixed to a horizontal beam above and passes over a second pulley fixed to the same beam down again to the man's hand.

If the man's weight together with the tackle is 150 lbs. what force must he exert to just support himself, the three portions of the string being parallel?

22. A quadrilateral $ABCD$ is capable of having a circle inscribed in it, and forces represented by BA, DA, DC, BC act on a rigid body; show that the resultant acts along the straight line passing through the centre of the circle and the middle points of the diagonals.

23. A uniform rod AB rests inclined at an angle α to the horizon with the end A on a rough horizontal plane and just about to slip, and with the end B supported by a string inclined at an angle β to the horizon. Prove that the coefficient of friction is $1/(\tan\beta - 2\tan\alpha)$.

24. A cannon-ball of mass m is shot from a gun of mass M (which is free to recoil in a horizontal direction) so that its muzzle-velocity relative to the ground is V. Show that its greatest range is $\dfrac{V^2}{g}$, and is obtained by giving the gun an elevation of $\tan^{-1}\left(1+\dfrac{m}{M}\right)$.

25. A stone is thrown from a given point with horizontal and vertical velocities u and v respectively; at the instant it reaches the highest point a second stone is thrown from the same point with horizontal velocity $3u$ so as to hit the first. Find what must be the vertical velocity of the second stone.

26. A light string has masses P and Q attached to its ends and is put over two fixed pulleys, the portion between them supporting a moveable pulley of mass R. Find the acceleration of P, all the portions of the string being vertical.

27. Two masses P and Q lie on a smooth horizontal table near each other and are connected by a string on which is threaded a ring of mass R. The ring hangs over the edge of the table, prove that it falls with an acceleration

$$\frac{R(P+Q)}{R(P+Q)+4PQ}g.$$

28. On a smooth wire bent into a circle and placed in a vertical plane slide two rings whose weights are as 1 to $\sqrt{3}$, the rings being connected by a light straight rod which subtends a right angle at the centre. Determine the positions of equilibrium.

29. A bullet weighing 1 oz. strikes a block of wood at rest with a velocity of 2400 ft.-secs. and remains imbedded in it, if the resultant velocity of the block and bullet is 16 ft.-secs. find the weight of the wood and the loss of kinetic energy.

30. A particle is sliding down the smooth face of a wedge whose other smooth face is in contact with a table on which it is free to move. If the angle of the wedge is 60° and its mass 4 times that of the particle show that its acceleration is $\frac{g\sqrt{3}}{19}$.

31. A fly-wheel whose mass of 66 lbs. is practically concentrated round a circle of 6 feet circumference is so mounted as to drive a machine that requires exactly $\frac{1}{10}$ of a H.P. to keep it going. The wheel is started with a speed of 40 revolutions a second. Assuming that the whole energy is absorbed at a uniform rate by the machine, find how long it will work.

32. A blow with an inelastic hammer-head of mass $\frac{1}{2}$ lb. drives a nail one and a half inches of its length into a board, the hammer-head being reduced to rest. If the velocity of the hammer-head when it first touched the nail was 15 ft.-secs., find the average resistance of the board to the nail's motion.

33. A 50-ton engine moving at the rate of 10 miles an hour impinges on a truck at rest weighing 10 tons and they both move on together, find their velocity and calculate the loss of kinetic energy.

34. Two equal heavy rods AB, BC are freely jointed at B and have their middle points connected by an inelastic string of half the length of either rod; if the system be suspended by a string attached to the end A, prove that the inclination of AB to the vertical will be $\tan^{-1}\frac{\sqrt{3}}{5}$.

Show also that the tension of the string will be $\frac{3}{\sqrt{7}}W$, and the stress at B will be equal to $\frac{2}{\sqrt{7}}W$, where W is the weight of a rod.

35. A locomotive is pulling a train of 10 carriages, each weighing 4 tons, up an incline of 1 in 100, the tension of the coupling between the two middle carriages is equal to 1000 lbs. wt., and the resistance due to friction, &c. is 16 lbs. wt. per ton. Find the acceleration of the train and the pull exerted on the first carriage by the engine.

36. The system of pulleys described as Case II. is modified by making the string which passes over each pulley A, pass round a small pulley attached to the weight, and the string is then fastened to the pulley A, the strings being all parallel. Show that if the weights of the pulleys be neglected and the number of strings be n, the mechanical advantage is $3^n - 1$.

37. From the lower end of the block of a pulley A is hung a weight W, and the upper end is supported by a string which after passing over a fixed pulley B supports a pulley C: another weight P is hung by a string which after passing over C and under A is tied to a fixed point. Find the condition of equilibrium, the pulleys being unequally heavy and the strings all parallel.

38. Two equal masses connected by an inextensible weightless thread that passes over a light pulley hang in equilibrium. Show that the tension of the thread is unaltered when $\frac{1}{n}$th of its mass is added to one and $\frac{1}{n+2}$th of its masses is removed from the other.

39. Two marksmen, using guns with the same muzzle velocity, fire at the same moment and simultaneously hit the same mark.

The one, who fires horizontally, is at the top of a tower of height h, and the other is on the ground. How far off from the tower is the latter, if the horizontal distance of the mark from the tower is d?

40. A mass of $\frac{1}{2}$ lb. falls from a height of 18 inches upon dough, which it penetrates to the depth of 3 inches; what is the average resistance exerted by the dough?

41. A weight W hangs by a string over a pulley. A monkey takes hold of the other end, and at an instant when W is at rest begins to climb and climbs a height h in t seconds without disturbing W. Determine his motion and find his weight.

If at the end of the t seconds he ceases to climb, how much further will he ascend in the next t seconds?

42. A shot whose mass is $\frac{1}{4}$ a ton is discharged from a gun whose mass is 110 tons, the gun's backward motion is checked by a constant pressure equal to the weight of 10 tons, and the recoil is observed to be $6\frac{1}{18}$ feet. Show that the muzzle-velocity of the shot is 1320 feet per second.

43. A man starts at right angles to the bank of a river at the uniform rate of $1\frac{1}{2}$ miles per hour to swim across; the current for part of the way is flowing uniformly at the rate of 1 mile per hour, and for the remainder of the way at double that rate. He finds when he has reached the other side that he has drifted down the stream a distance equal to the breadth of the river. At what point did the current change?

44. A heavy board in the form of a right-angled triangle ABC is suspended at the right angle C. If two equal particles starting simultaneously from C slide down the sides CA, CB, show that they will reach A, B at the same moment, and that the motion will not disturb the equilibrium of the triangle.

45. A table with a heavy rectangular top $ABCD$ rests upon 4 equal and equally heavy legs placed at A, B, E, F where E and F are the middle points of BC and CD. Show that the table will be upset by a weight placed upon it at C just greater than the weight of the whole table, and find the greatest weight that may be placed at D without upsetting the table.

46. An endless string without weight of length l hangs in two loops over two smooth pegs in the same horizontal line at a distance a apart, and on each loop is placed a small smooth ring, one of weight W and the other of weight W'. Find an equation giving the tension of the string.

47. By the principle of work show that the pull of a locomotive engine is pd^2l/D lbs., for a mean effective pressure p lbs. on the square inch, where d is the diameter of each of the cylinders, l the length of the stroke, D the diameter of the driving-wheel of the engine.

48. Show that the mechanical advantage of the pulley in the cases I. and II. (p. 175) are reduced to $(1+m)^n$ and $(1+m)^n-1$, when it is found that the friction of the ropes causes the tension to be reduced to m times its value in passing round a pulley.

49. Two equal rods AC, BC, each of weight W, are hinged together at C and placed to stand with A, B on a rough horizontal plane. If $AB=2a$ and $h=$height of C above the plane, prove that if they are just in equilibrium $\mu=\dfrac{a}{2h}$.

Also if μ exceed this value show that a weight $=W\dfrac{2\mu h-a}{a-\mu h}$ might be placed at C without slipping taking place at A or B.

50. An elastic ball of mass m moving on a smooth horizontal plane impinges on a ball of equal size but of different mass m' which is at rest on the plane. If just before impact the line of motion of the impinging ball be inclined at the angle a to the line joining the centres of the balls, prove that the direction of the impinging ball will be turned through a right angle if $m=m'(e\cos^2 a-\sin^2 a)$.

51. Two elastic spheres equal in all respects are moving towards each other with equal velocities, their centres being on two parallel lines whose distance apart is d_1. Prove that after impact they will move away from each other with equal velocities, so that their centres are on two parallel lines whose distance apart d_2 is given by

$$d_2^2\{e^2d^2+(1-e^2)\,d_1^2\}=d^2d_1^2,$$

where d is the diameter of either sphere.

52. Two equal ivory balls are suspended in contact by two equal parallel strings so that the line joining the centres of the ball is horizontal and 2 feet below the points of attachment of the threads. Find

the coefficient of elasticity when it is found that allowing one ball to start from the position when its thread makes an angle of 60° with the vertical causes the other ball after impact to come to rest in a position where its centre is 1 foot 8 inches from its original position of equilibrium.

53. A particle whose mass is 3 ozs. revolves on a smooth horizontal table, being attached to a fixed point by a light string of length 10 feet. If the greatest tension the string can bear is equal to 1 cwt., find the velocity of revolution required to break the string.

54. If an elastic ball be reflected in succession by each of two smooth vertical planes at right angles in a horizontal plane, show that its directions before the first impact and after the second are the same.

55. If the unit of time is 1 minute, the unit of length 1 mile and the unit of mass the mass of a ton, what unit of force is implied in the equation $F=ma$?

56. The unit of work is the work done in raising 50 tons through 20 feet, the unit of acceleration is 16 ft. secs. per second, the unit of density that of a substance of which a cubic yard weighs 2 cwt. Find the units of length, time, mass and force.

57. If an hour be the unit of time and 4000 miles the unit of length, find the value of g.

58. If in a frictionless wheel and axle a force of 10 lbs. wt. supports a load of 120 lbs. wt., find the acceleration of the load when 10 lbs. wt. is taken from it and added to the force.

59. A truck whose mass is m tons is drawn from rest by a horse through a feet and is then moving at the rate of b feet per second. If the resistances are equivalent to c lbs. wt. per ton, show that the work done by the horse is $35mb^2+mac$ foot-lbs.

60. A ball of mass m_1 strikes a ball of mass m_2, which is at rest, with velocity u. Both balls are free to move inside a smooth horizontal circular tube, prove that after n impacts the K.E. of the system is $\frac{1}{2}m_1u^2\left(\frac{m_1+m_2e^{2n}}{m_1+m_2}\right)$, where e is the coefficient of elasticity.

61. A uniform rod of length $2a\sin\alpha$ rests within a rough vertical circle of radius a, show that the greatest possible inclination of the rod to the horizon is $\tan^{-1}\left(\frac{\mu}{\cos^2\alpha-\mu^2\sin^2\alpha}\right)$.

62. A picture-frame rectangular in shape rests against a smooth vertical wall from two points in which it is suspended by parallel strings attached to two points in the upper edge of the back of the frame, the length of each string being = the height of the frame.

Show that the picture will rest against the wall at an angle $\tan^{-1}\frac{b}{3a}$, where a is the height of the frame, and b its thickness.

63. A smooth wedge of angle α is free to slide along a smooth horizontal table in a direction perpendicular to the edge of the wedge. On the surface of the wedge move two particles of masses m and m' connected by a fine inextensible string which passes round a smooth peg driven into the wedge. The two straight portions of the string lie along lines of greatest slope. If M is the mass of the wedge, prove that the tension of the string is

$$\frac{2mm'(M+m+m')g\sin\alpha}{M(m+m')+4mm'+\sin^2\alpha(m-m')^2}.$$

64. If a pendulum fits loosely on a horizontal axis of radius a and is found to make a constant inclination θ to the horizon when the axle is kept rotating, the angle of friction ϕ between the rubbing surfaces is given by $\sin\phi=\frac{b}{a}\sin\theta$, where b is the distance of the c.g. of the pendulum from the point of suspension.

65. Show that a railway carriage running round a curve of radius r will upset if the velocity is greater than $\sqrt{\frac{gra}{2h}}$, where a is the distance between the rails, and h the height of the c.g. of the carriage above the rails.

66. The sides of a triangular framework are 13, 20 and 21 inches long, the longest side rests on a horizontal smooth table and a weight of 63 lbs. is suspended from the opposite angle. Find the stress in the side on the table.

67. If four forces in one plane be in equilibrium and the lines of action of all be given but the magnitude of only one, show how the magnitudes of the other three may be determined by the graphical method. Four heavy rods equal in all respects are freely jointed together at their extremities so as to form the rhombus $ABCD$. If this rhombus be suspended by two strings attached to the middle points of AB and AD, each string being inclined at an angle θ to the vertical, the angles of the rhombus will be 2θ and $\pi-2\theta$.

68. Two equal heavy rods AC, BC are jointed together at C and have their other extremities A and B jointed to fixed pegs in the same vertical line. Prove that the direction of the stress at C is horizontal and determine by a geometrical construction the stresses at A and B.

If a weight which is equal to that of either rod be attached to the centre of the lower rod, show that if α is the inclination of each rod to the vertical and θ the inclination to the vertical of the stress at C $\tan\theta=3\tan\alpha$, and that this stress is to the weight of either rod in the ratio $\sqrt{1+9\tan^2\alpha} : 1$.

69. Three equal uniform rods, each of weight W, are joined to form an equilateral triangle. If the system be suspended by the middle point of one of the rods the stress at the lowest angle is $\frac{W}{2\sqrt{3}}$, and that at each of the upper angles is $\frac{1}{2}W\sqrt{\frac{13}{3}}$.

70. Three uniform rods whose weights are proportional to their lengths are freely jointed together to form a triangle ABC which is placed with its plane vertical and its side BC on a horizontal plane. Show that θ, the inclination of the stress at A to the horizon, is given by the equation

$$\tan B \sin (B-\theta) = \tan C \sin (C+\theta).$$

ANSWERS TO THE EXAMPLES

Ex. I. Page 4.

1. 88. 2. $\frac{1}{4}$ min. 3. [illegible] miles per minute.
4. 26,400 feet. 5. 40. 6. 8·15 minutes nearly. 7. 2 miles.

Ex. II. Page 7.

1. $a=8$. 2. 4 secs. 3. 8.
4. $v=180$. 5. 11 secs. 6. 15 feet per second.

Ex. III. Page 12.

1. 1600 ft., 320 ft. per sec. 3. 6 ft. per sec. 4. 10 secs.
5. 100 secs., 10,000 cms. 6. 1 min.

Ex. IV. Page 13.

1. 440 ft. 2. 27 ft.-sec. units. 3. 1 sec., 16 ft.
4. $u=15$, $a=-\frac{1}{2}$.

Ex. V. Page 18

1. 80 ft.-sec. units. 2. 510 ft., 2430 ft.
3. The distances are as 11 : 23. 4. $u=10$, $a=2$. 5. 3 secs., $4\frac{1}{2}$ ft.

Ex. VI. Page 23.

1. 256 ft., 64 ft. per sec. 2. 32 ft. 3. $22\frac{1}{2}$ ft.
4. 405 ft. nearly, 177 ft. 5. 136 ft. per sec., 88 ft. 6. 80 ft.
7. 144 ft. 9. 4080 ft. 10. 861 ft., 464 ft. per sec.
11. 1200 ft. per sec. 12. $85\frac{1}{3}$ ft.-secs. 15. 11,000 feet.
16. 20 miles per hr. 20. 9 secs.
21. Vel. of train : vel. of particle as 3·9 : 1 nearly.

Ex. VII. Page 35.

1. 28 poundals. **2.** 16 ft. **3.** $234\frac{2}{3}$ secs. **4.** 36,000 ft.
5. 141 ft.-secs. **6.** 1010 lbs. wt. **7.** 3200 units of momentum.
8. 200 poundals, $\frac{10}{112}$ ft.-sec. units. **9.** ·017 nearly, 855·5 secs.
10. ·0086. **11.** 5 lbs. **12.** $36\frac{1}{3}$ cm. per sec., $18\frac{1}{6}$ cm.

Ex. VIII. Page 39.

1. 980 in c.g.s. units. **2.** 52 ft., in $6\frac{1}{2}$ sec. **3.** $\frac{1}{4}$ sec., 1 foot.
4. 36 feet.

Ex. IX. Page 42.

1. $\frac{1}{32}$ of a ton weight. **2.** 6 ft.-sec. units.
3. (i) $\frac{5}{16}$ cwt.; (ii) [illegible] cwt. **4.** (i) 40 lbs. wt.; (ii) 50 lbs. wt.
5. Zero. **6.** 5 sec., 80 ft.-sec.
7. Acceleration = 4 ft.-sec. units, tension = wt. of $\frac{63}{8}$ oz.; pressures = $4\frac{1}{2}$ and $5\frac{1}{4}$ oz. wt.
8. The forces are equal. **9.** (i) 4·7; (ii) 234·4 nearly.
10. 3348 tons wt. nearly. **11.** ·0174 sec. nearly.
12. True weight = 8·9 lbs. **13.** 3·7 sec., 17 ft.-secs.
16. $24f + 23f' = g$. **18.** 6·5 ft.-secs., 7·3 ft.-secs.
21. $R - 3PQ/(P + 4Q)$. **22.** 17·3 ft. nearly.

Ex. X. Page 51.

1. (i) 2·9; (ii) 5·5; (iii) 14·2; (iv) 19·1. approx.
3. 50 ft./sec. **4.** 17·3 nearly. **5.** 30° E. of S.
6. Vel. = $88/15\sqrt{3}$ ft. per sec., width of deck = 40·6 ft.
8. (i) 15·5 lbs. wt. approx.; (ii) 1·5 lbs. wt. approx.; (iii) 3 lbs. wt.; (iv) 3·2 lbs. wt. approx.
10. 1 lb. wt., $\sqrt{7}$ lbs. wt. **11.** 2 lbs. wt.
12. $2 + \sqrt{3}$, and $2 - \sqrt{3}$ lbs. wt. **13.** 18 lbs. nearly. **14.** 5 : 3.
15. $Q \sim Q'$. **17.** 5·8 lbs. making an angle of 30° with the vertical.
18. $\sqrt{2} : 1$. **19.** $12\sqrt{2}$ lbs., 12 lbs. **21.** $-\frac{1}{5}$.
22. $\frac{20}{3}$ lbs., $\frac{10}{3}\sqrt{5}$ lbs. **24.** $\frac{24}{25}$.

Ex. XI. Page 57.

1. 4 lbs. **2.** A force represented by $\sqrt{3}$.
3. Tensions = 6 and 8 lbs. wt. **5.** 120°. **6.** $P \sim Q$.
7. $2(\sqrt{2} - 1)$ lbs. wt. due East. **10.** 20 and $10\sqrt{3}$ lbs. wt.
11. [illegible], [illegible]. **12.** 1·46 and 4·7 tons wt.

Ex. XII. Page 64.

1. $15\sqrt{3}$ ft.-secs., 15 ft.-secs. 2. 28·3 ft.-secs. nearly. 4. 100 ft.
5. 9, 37·4, approx. 6. 6·5 ft. per min. nearly.
7. 5 and 5·4 ft.-secs. 8. 138 ft.-secs. nearly.

Ex. XIII. Page 67.

1. 4·8 lbs. wt. 2. 4 lbs. wt. North. 3. $\sqrt{3}F$. 4. 4 poundals.
5. 8·29 lbs. 10. The resultant is rep. by AB. 13. CA.
15. $\sqrt{P^2+Q^2+R^2+S^2-2PR-2QS}$.
16. $\sqrt{P^2+Q^2+R^2-QR-RP-PQ}$. 17. $\sqrt{3}:1:2$.
18. 135°, 135°, 90°. 19. 3 lbs., 120°, 30°.
20. $\frac{3\sqrt{3}}{2\sqrt{37}}$ with 4, $\frac{2\sqrt{3}}{\sqrt{37}}$ with 3. 21. 30°.

Ex. XIV. Page 69.

1. 25 miles per hour. 2. 35 miles. 3. 94 ft.-secs. nearly.
4. 30 miles per hour. 5. 10 miles per hr. from N.W. 7. $v \cot \theta$.
8. If α is the angle its direction makes with the line of wickets, $\tan \alpha = \frac{3}{4}$.
9. He walks in a direction making an angle $\sin^{-1}\frac{3}{5}$ with his rank, in $\frac{135}{8}$ secs. 10. 12 secs. 11. 9 minutes; $\sqrt{5}$ miles.

Ex. XV. Page 72.

1. 1 lb. wt.
3. If A is the angle through which the force is turned, the resultant makes an angle $90° - \frac{1}{2}A$ with the direction of this force produced.
4. 14·1 tons wt. nearly. 5. 50 ft./sec. nearly. 7. $\frac{1}{10}$ ton wt.
10. The shorter. 11. 17 lbs. wt. 12. 120°.
16. If P is the given force, each component is $\frac{\sqrt{10}}{6}P$. 17. 30°.

Ex. XVI. Page 78.

1. 5040. 3. $79\frac{1}{16}$ ft.-lbs. 4. $50m$ ft.-poundals. 5. 22,400 ft.-lbs.

Ex. XVII. Page 82.

1. 75 miles per hour. 2. $6\frac{6}{35}$. 3. 30·72. 4. $130\frac{2}{15}$.

Ex. XVIII. Page 84.

1. $1\frac{13}{18}$ feet. 2. 48,020,000 ergs; 12,005,000 ergs; 0. 3. $\frac{125}{32}$ feet.
4. 18×10^{10} ergs. 5. 150 and 7500 poundals. 6. $11·6 \times 10^5$ joules.

Ex. XIX. Page 88.

1. ·94 ft.-lbs. 2. 3·7 ft.-lbs. 4. 25 ergs.

Ex. XX. Page 91.

1. 1200. 2. $\frac{440}{11}$. 3. Unit of length $= \frac{2}{5}$ feet, unit of time $= \frac{2}{5}$ sec.
4. $3^3 \times 20^4$. 5. ·00007, ·0000024 approx. 6. 2240 lbs., 800 ft., 5 sec. 7. $\frac{11}{10800}$ foot-secs. 10. $3332\frac{24}{121}$ lbs.

Ex. XXI. Page 92.

1. 179·2 H.P. 2. 2·52 ft.-lbs. 6. $\sqrt{3}$ lbs. 7. 4320.
9. 30 H.P. 12. 1·8 secs. approx. 13. 2 secs.
15. 9·6 inches. 16. 40 ft. per sec.

Ex. XXII. Page 104.

1. (i) 35 lbs. wt. distant $8\frac{4}{7}$ inches from the greater force;
(ii) 5 ....60 ... .
2. 4 inches. 3. $22\frac{1}{2}$, $7\frac{1}{2}$ lbs. weight. 4. 40 lbs., 120 lbs.
5. $4\frac{2}{13}$. 6. The distances are as $b : a - b$.
7. If W is the weight of the bundle, and a, x, the distances of the bundle and his hand from his shoulder, the pressure is $W\left(1 + \frac{a}{x}\right)$.
8. $16\frac{1}{2}$ inches, $13\frac{1}{2}$ inches.

Ex. XXIII. Page 107.

1. 15 inches 2. 16 lbs., 48 lbs. wt. 3. Where the force 9 acts.
4. 8 inches from the middle point. 5. In the middle.
9. $50\frac{l}{l+6}$ lbs. wt., where l is the length of the rod in inches.

Ex. XXIV. Page 111.

1. $6\frac{2}{13}$ feet from the boy. 2. As 1 : 5. 3. $4\frac{1}{2}$, $7\frac{2}{3}$ feet.
4. The required distance is equal to the radius of the inscribed circle of the triangle.
7. 40 lbs. weight, parallel to BC and distant 3 feet from BC.
11. It is represented by CD, where D is a point in AB such that
$$AD : DB :: 7 : 5.$$
15. The other forces are in the directions CB, CD, AD.

Ex. XXV. Page 120.

1. A force $2\sqrt{2}$ in one of the diagonals. 2. A force 6 lbs. in a line parallel to the 11 lb. force and distant $1\frac{1}{4}$ times the side of the square.

Ex. XXVII. Page 127.

1. The c.g. is 6 and 30 inches from the respective ends.
2. $1\frac{4}{11}$ inches from the point of contact. 3. 13 inches, $2\sqrt{34}$ inches.
4. $\frac{1}{181}$ inch from the centre of the rod.
5. (i) $\frac{3}{31}$ inch from the centre.
   (ii) $5\frac{5}{11}$ inches from the centre of the larger plate.
   (iii) $\frac{1}{15}$ inch from the centre of the square.
   (iv) $\frac{1}{2}$ inch from the centre of the rod.

Ex. XXVIII. Page 128.

1. The midpoint of the middle weights.
2. $3\frac{1}{3}$ inches from the midpoint.
3. $2\frac{3}{13}$ inches. 4. $3\frac{6}{13}$ feet. 5. 9 lbs. 6. $8\frac{13}{33}$ in.

Ex. XXIX. Page 131.

1. The c.g. is $\frac{33}{56}a$, $\frac{17}{56}a$ distant from the sides which meet where the weight 2 is placed, a being a side of the square.
2. $\frac{2a}{\sqrt{3}}$, if a is a side. 3. $\frac{a}{3}$ from the middle side.
4. The c.g. lies on the production of the line joining the centre to the vertex at which there is no particle and at a distance $\frac{1}{8}r$, where r is the radius of the circumscribing circle. 5. $\frac{9\sqrt{3}}{14}a$ from the first side.
6. $\frac{26}{27\sqrt{2}}a$ from the vertex, where a is a side.
7. 2 inches. 8. $\frac{21}{8}$ inches from the bottom.

Ex. XXX. Page 136.

1. $\frac{a}{6\sqrt{2}}$ from the centre of the square, if a is a side. 2. $r=1+\sqrt{2}$.
3. The c.g. is distant $\frac{7}{11}a$, $\frac{5}{11}a$ from the sides through the first vertex.
4. The centre of the inscribed circle.
5. O is the c.g. of the triangle. 6. $2\frac{1}{11}$ in. from the angle.
8. If $\bar{x}, \bar{y}$ are the distances of the c.g. from the middle point of one side $2a\bar{y}=3\bar{x}^2$. 10. $7\frac{1}{11}$ in., $8\frac{3}{4}$ in., from two edges.
13. $5\frac{1}{3}$ inches. 15. The c.g. bisects the rod, which weighs 12 lbs.
21. On the base of the triangle. 22. $\frac{R^2+r^2+Rr}{R+r}$, $\frac{3R}{2}$.
23. 660,000 ft.-lbs.; 30 H.P. 24. 10·7 nearly. 25. 3·2 H.P.
27. About $\frac{1}{4}$ of the volume. 29. $\sqrt{5}-1:2$.

Ex. XXXI. Page 152.

1. The diagonal through the point is inclined to the vertical at an angle α such that $\tan\alpha = \frac{1}{3}$.
2. Tension $= 10$ lbs. wt., pressure $= 6$ lbs. wt.
3. Required weight $= 6 \cdot 6$ lbs.
6. Each is a force of 3 lbs. wt.
7. Thrust $= \frac{1}{2} W \cot \frac{A}{2}$.
8. To a point in the same horizontal plane as the centre.
9. Pressure $= W$.
10. Two ozs.
11. $W \dfrac{r}{2\sqrt{R^2 - r^2}}$.
13. $\tan^{-1}\left(\dfrac{\sin\alpha}{\sin\alpha + 2\cos\alpha}\right)$.
17. $15a$.
23. The angle between the strings is $2\cos^{-1}\frac{1}{24}$.
28. 620 lbs.; 730 lbs.

Ex. XXXII. Page 160.

1. $1\frac{3}{7}$ lbs. wt.
2. 5 ft. and 2 ft. from the ends.
3. Fulcrum is $2\frac{1}{3}$ ft. from the centre.
4. 170 lbs. wt.
5. $9\frac{3}{8}$ lbs. wt.
6. The arms are as 9 : 11.
7. Shorter wire is inclined to horizon at the angle $\cot^{-1} 2\sqrt{3}$.
8. $13\frac{1}{3}$ feet, 15 lbs.
9. $62\frac{1}{2}$ lbs. wt.
10. 50 lbs. wt., neglecting weight of lever.

Ex. XXXIII. Page 163.

1. $26\frac{1}{4}$ lbs.
2. 56 ozs.
3. 2 : 1, 8 lbs.
7. He loses £$19 \times \dfrac{(a-b)^2}{ab}$.

Ex. XXXIV. Page 166.

1. 44 inches.
2. $2\frac{1}{2}$ inches.
4. If O' be the new zero of graduation $CO' = \frac{9}{10} CO$.
5. One inch from one end.
6. 18 lbs.

Ex. XXXV. Page 168.

1. 3 ozs.
2. 20 inches.
3. $\frac{17}{7}$.
4. 2 inches.
5. Fix on a weight so that the c. g. and the mass are both the same as before.

Ex. XXXVI. Page 171.

1. $4\frac{4}{5}$ lbs. weight.
2. 120 lbs. wt.
3. 12, 36 lbs. wt.
4. $3181\frac{5}{7}$ lbs. wt.
5. $3\frac{4}{7}$ inches.
6. $2\frac{4}{7}$ lbs. wt.
8. 300 lbs. wt.
9. $32\frac{3}{8}$ lbs. wt., $\frac{14 \times 14}{3 \times 550}$ H.P.

Ex. XXXVII. Page 175.

1. 12 lbs. wt.
2. $4P$.
3. 2 lbs. wt., 7 pulleys.
4. 207 lbs. wt.
5. $\frac{31}{32}$ of the man's weight.
7. 9 stone weight, neglecting the weight of the pulleys.
8. The radii of the wheels in the upper block are in the proportion 2 : 4 : 6 &c., and those in the lower block in the proportion 1 : 3 : 5 &c.

Ex. XXXVIII. Page 180.

1. 14 lbs. wt.
2. $15\frac{3}{4}$ lbs. wt.
3. 290 lbs. wt.
4. 1 lb. wt.
6. 4 lbs. wt.
7. Unity.
8. 97 lbs. wt., 1 lb. wt.
9. 70 lbs. wt.
10. $71\frac{1}{4}$ lbs. wt.
12. $14\frac{2}{3}$ lbs. wt.
13. 62 lbs. wt.
14. 36·2.
16. The distance from the point of action of T_1 is $\frac{1}{7W}\{20W + 10w_1 + 8w_2\}$.

Ex. XXXIX. Page 184.

1. 12 feet.
2. 1 cwt.
3. 9 lbs. wt., 15 lbs. wt.
4. 12 lbs. wt.
5. $\sqrt{24}$, 5.
7. $5\frac{1}{3}$ lbs. wt.
8. 60°.
9. 84 lbs. wt.

Ex. XL. Page 186.

1. $20\sqrt{3}$.
2. 60°, 120° with the plane.
3. $\cos^{-1}\frac{1}{3}$, $2P$.
5. 120° with inclined plane.

Ex. XLI. Page 190.

1. $2\frac{4}{7}$ lbs. wt.
2. $13\frac{4}{11}$ lbs. wt.
3. $\frac{2}{15}$.

Ex. XLII. Page 199.

1. ·066.
5. It makes the angle $\tan^{-1}\frac{15}{8}$ with the horizon.
6. $\sqrt{2}-1$, $\frac{W}{2\sqrt{2}}$.
7. 3808 ft.-lbs.
10. At the middle point of the edge vertically above the one which moves off the floor.

Ex. XLIII. Page 203.

1. $\frac{l'}{l} = \frac{t'}{t} = \frac{\sin(\alpha - \epsilon)}{\sin(\alpha + \epsilon)}$.

10. 3,097,600 sin λ foot-lbs.

15. 27·4 lbs. approx.

Ex. XLIV. Page 209.

1. 300 ft., 456 ft. 2. 10,000 ft.; 20,000 ft. 3. 96 ft.

4. 12,100 yds. 6. 2700 yds.; 14,400 yds.

Ex. XLV Page 212.

1. 45, $40\sqrt{6}$ ft. secs. 2. 173 ft.-secs. nearly.

3. $30\sqrt{10}$ ft.-secs. 15. $\pi r^1/g^2$ sq. feet.

Ex. XLVI. Page 220.

1. $\frac{8}{3\pi} = \frac{28}{33}$ nearly. 2. $\frac{4}{5\pi}$. 3. When the radius makes an angle $\cos^{-1}\frac{2}{3}$ with the vertical.

4. $8\sqrt{r}$, where r = radius of circle.

5. $2\frac{1}{10}$ cwt.

9. Momentum of each $= 12^2 \times 112$ units; K.E. of cannon = 3024 ft.-lbs.; K.E. of ball $= 9 \times 112^2$ ft.-lbs.

10. ·107 lbs. wt. 14. $\sqrt{(\frac{2}{3}ga\cos\alpha)}$. 16. 1·17 approx.

18. The particle leaves circle at an angular distance $\cos^{-1}\frac{2}{3}$ from the highest point of circle, and joins it again at an angular distance $3\cos^{-1}\frac{2}{3}$.

19. $7{\cdot}4m$, $9{\cdot}9m$ lbs. wt. approx.

Ex. XLVII. Page 228.

1. Pendulum must be lengthened $2/15\pi^2$ inch = ·014 inch nearly.

2. 3692 nearly. 4. 9 secs. nearly. 5. $5\pi^3/64$ ft.-lbs., or ·77 ft.-lbs. approx.

6. Distance $= 2lWa/\lambda$; time $= 2\pi\sqrt{(Wa/g\lambda)}$.

7. If $2l$ = distance between fixed points, and a = distance through which particle is displaced, period $= \pi\sqrt{(ml/\lambda)}$, and greatest velocity

$$= 2a\sqrt{(\lambda/ml)}.$$

Ex. XLVIII. Page 237.

1. $\frac{3}{8}u$, where u is the vel. of the striking sphere.

3. $e = \frac{2}{3}$; impulse $= \frac{35\sqrt{75}}{4}$. 4. As 1 : 50.

10. 9·7 sec., 45 sec. nearly.

15. The vel. of each ball is $\frac{\sqrt{48}}{8}v$, and makes an angle α with the line joining their centres where $\tan\alpha = 4/3\sqrt{3}$.

...ance = $81\frac{1}{4}$ cwt.

22. e = ·998.

Ex. XLIX. Page 251.

1. $5\sqrt{3}$, $5\sqrt{3}$ and $\frac{5}{2}\sqrt{3}$ lbs. wt.
2. Supporting forces $= 11\frac{3}{8}$ tons, $22\frac{5}{8}$ tons; shearing forces $= 11\frac{3}{8}$, $6\frac{3}{8}$, and $-4\frac{5}{8}$ tons; bending moments $= 56\frac{1}{8}$, 86, and $82\frac{7}{8}$ tons-feet.

Ex. LI. Page 259.

1. Moment of inertia $= \frac{1}{2}M(r_1^4 - r_2^4)/(r_1^2 - r_2^2) = \frac{1}{2}M(r_1^2 + r_2^2)$; hence kinetic energy $= 10{\cdot}3 \times 10^5$ ft.-lbs. approx.
2. 543·3 lb.-ft.^{2} units, approx.
3. $g/16$.
4. ·78 sec. approx.
5. 1·41 secs. approx.
7. ·54 of a penny.

MISCELLANEOUS EXAMPLES

Page 260.

1. $13\frac{7}{11}$, $27\frac{3}{11}$ miles per hour.
2. $8\sqrt{3}$ miles per hour; 15° to W. of S.
4. $\frac{11}{16}$ of a mile.
8. $\left\{28\dfrac{M - M'\sin a}{M + M'}\right\}^{\frac{1}{2}}$.
11. 2·49 feet.
17. $Q \,.\, AB + R \,.\, AD = 1$.
21. 50 lbs.
25. Vertical velocity $= v$.
26. Acceleration of $P = \dfrac{P \,.\, R - 3Q \,.\, R + 4P \,.\, Q}{(P+Q)R + 4PQ}$.
29. 9·3125 lbs., $99\frac{1}{3}$ per cent. K.E. is lost.
31. 18 minutes.
32. 450 poundals.
33. $8\frac{1}{3}$ miles per hour; $\frac{1}{3}$ of the K.E. is lost.
35. $\frac{3}{125}$; 2000 lbs.
37. $4P = W + A - C$.
38. $d \mp \sqrt{d^2 - h^2}$.
40. $8\frac{1}{2}$ lbs. wt.
41. $\dfrac{Wg}{g + 2h/t^2}$, $\dfrac{hgt^2}{2gt^2 + h}$.
43. Three-quarters distance across.
45. $\frac{1}{2}W$.
46. $\dfrac{l}{a} = \dfrac{1}{\sqrt{1 - \left(\dfrac{W}{2T}\right)^2}} + \dfrac{1}{\sqrt{1 - \left(\dfrac{W'}{2T}\right)^2}}$.
52. $\frac{2}{3}$.
53. 7 revolutions per second nearly.
55. $102\frac{2}{3}$ lbs.
56. $224000 \times \sqrt{\frac{2}{3}}$ poundals.
57. $19\frac{1}{11}$.
58. $\frac{1}{16}g$.
66. $6\frac{2}{3}$ lbs.